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## Studies on the effect of some-agro-chemical on nitrogen transformations in soil 1—Nitrate nitrogen carrier

By

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### Abstract

Effect of telodrin and gamma isomer of benzene hexachloride on transformations of soil and fertilizer nitrogen under two moisture regimes have been investigated. These pesticides were found to accelerate the accumulation of available nitrogen interlocked in soil organic matter probably by increasing the activity of microorganisms responsible for mineralisation of soil organic matter. Telodrin at higher rate of application also favoured accumulation of ammoniacal nitrogen. With the addition of fertilizer nitrogen in the form of sodium nitrate, gradual loss of nitrogen was noticed which was suppressed to a great extent with the pesticidal treatments.

Soil insecticides like telodrin\* and gamma BHC\*\* when applied to crops through soil have been reported to be very efficacious in the control of soil harbouring pests and boosting yields<sup>6,8</sup>. Yield increase by these chemicals especially the former has been found to be of such a magnitude that it can hardly be ascribed to control of pests alone. In search of other probable factors responsible for this yield boost a series of experiments were conducted. Babu *et al*<sup>1</sup> reported that by application of these chemicals nitrogen concentration in 3<sup>rd</sup> 6<sup>th</sup> leaf lamina of sugarcane increases. *A Priori* it was suspected that these chemicals may in some-way be favourably affecting the utilization of nitrogen in soil. In this communication the effect of these two chemicals on transformations of soil and sodium nitrate fertilizer nitrogen under controlled laboratory conditions have been reported. Cane crop requires frequent irrigation and thus the moisture saturated condition is seldom encountered. In view of this fact the studies have been conducted under two moisture levels *i.e.* 40% water holding capacity and 100% water holding capacity.

\*Telodrin is Shell chemicals' trade name for 1, 3, 4, 5, 6, 7, 8, 8-Octachloro-3  $\alpha$ -4- tetrahydro-4, 7, methanonaphthalan.

\*\*Gamma BHC is gamma isomer of 1, 2, 3, 4, 5, 6- hexachlorocyclohexane

## Material and Methods

Soil used in these experiments was collected from the fields of Sugarcane Research Station, Jullundur Cantt. It was dried in air under shade and passed through 100 mesh sieve. For incubation work 500 g lots of this soil were taken in each of the required number of wide mouthed glass bottles and treatments shown in table I and II were set up. Nitrogen was added in the form of sodium nitrate (laboratory grade) at the rate of 100 ppm. Telodrin (15% E. C.) and gamma BHC (20% E. C.) were first emulsified with water and then required quantity added to the bottles. Additional amounts of water were added to the soil contained in bottles to obtain in requisite moisture level. Now the contents of the bottle were thoroughly mixed to ensure uniform distribution to additives. To maintain required moisture level during the incubation period distilled water was added every five days to make up for evaporation losses and contents were thoroughly mixed. The experiment was run in two series. Mean average temperature during the course of the experiment was 28°C.

TABLE I  
*Effect of pesticides on nitrogen transformation in soil*  
Moisture level—40% water holding capacity

Treatments	Ammonical nitro- gen ppm				Nitrate nitrogen in ppm				Amm.+nitrate nitrogen ppm				Total nitrogen ppm	
	Period in days				Period in days				Period in days				Period in days	
	0	10	30	75	0	10	30	75	0	10	30	75	30	75
Soil alone	8	6	6	4	12	16	20	18	20	22	26	22	337	323
Soil + T <sub>1</sub>	6	10	5	8	16	20	24	26	22	30	29	34	337	326
Soil + G <sub>1</sub>	5	4	7	8	14	20	22	23	19	24	29	31	332	323
Soil + T <sub>2</sub>	8	16	20	11	15	22	27	27	23	38	47	38	342	331
Soil + G <sub>2</sub>	9	6	12	10	10	26	20	25	19	32	32	35	339	328
Soil + NaNO <sub>3</sub>	9	8	10	8	109	102	92	84	118	110	102	92	432	408
Soil + NaNO <sub>3</sub> + T <sub>1</sub>	6	14	12	9	113	106	102	100	119	120	114	109	438	423
Soil + NaNO <sub>3</sub> + G <sub>1</sub>	5	10	7	9	112	110	100	93	117	120	107	102	427	410
Soil + NaNO <sub>3</sub> + T <sub>2</sub>	10	15	15	14	114	108	105	100	124	123	120	114	435	428
Soil + NaNO <sub>3</sub> + G <sub>2</sub>	10	10	15	10	116	102	106	98	126	112	121	108	430	415

T<sub>1</sub>, T<sub>2</sub> represent rate of application of telodrin at 0.75 and 3.75 ppm (actual ingredient) respectively.

G<sub>1</sub>, G<sub>2</sub> represent rate of application of gamma BHC at 1 and 5 ppm (actual in gredient) respectively.

TABLE II  
Effect of pesticides on nitrogen transformations in soil  
Moisture level—100% water holding capacity

Treatments	Ammoniacal nitro- gen ppm				Nitrate nitrogen in ppm				Amm.+nitrate nitrogen ppm				Total nitrogen ppm	
	Period in days				Period in days				Period in days				Period in days	
	0	10	30	75	0	10	30	75	0	10	30	75	30	75
Soil alone	7	7	8	7	10	15	15	16	17	22	23	23	331	319
Soil + T <sub>1</sub>	7	12	10	14	14	16	20	20	21	28	30	34	331	323
Soil + G <sub>1</sub>	6	5	11	10	12	20	16	20	17	26	27	30	332	326
Soil + T <sub>2</sub>	11	14	25	20	6	12	16	24	17	26	41	44	332	328
Soil + G <sub>2</sub>	8	9	16	10	16	16	24	24	24	25	40	34	336	321
Soil + NaNO <sub>3</sub>	10	4	12	9	112	100	84	75	122	104	96	83	420	390
Soil + NaNO <sub>3</sub> + T <sub>1</sub>	8	13	16	10	112	107	96	89	120	120	112	99	430	411
Soil + NaNO <sub>3</sub> + G <sub>1</sub>	10	7	13	10	106	112	98	85	116	119	111	95	420	400
Soil + NaNO <sub>3</sub> + T <sub>2</sub>	10	14	25	18	115	106	98	90	125	120	123	108	435	420
Soil + NaNO <sub>3</sub> + G <sub>2</sub>	9	12	13	10	108	107	86	94	117	119	99	104	430	410

T<sub>1</sub>, T<sub>2</sub> represent rate of application of telodrin at 0.75 and 3.75 ppm (actual ingredient) respectively.

G<sub>1</sub>, G<sub>2</sub> represent rate of application of gamma BHC at 1 and 5 ppm (actual ingredient) respectively.

At intervals of 10, 30 and 75 days soil samples were drawn and various determinations made. For zero days reading, samples were drawn 16 hours after starting incubation to make allowance for stabilization of moisture and additives. While drawing soil samples for other determinations, 15 g soil was simultaneously taken in china clay dishes and heated for 8 hours in an electric oven at 100–110°C. and moisture percentages calculated. After making necessary corrections for moisture, results were expressed on oven dry basis. Ammoniacal nitrogen was determined in N KCl leachate of moist soil by distilling with ignited MgO, ammonia evolved being absorbed in 4% boric acid and titrated with 0.02 N H<sub>2</sub>SO<sub>4</sub>. For nitrogen, the residue left after ammonia determination was diluted with water, reduced with Devarda's alloy, distilled, collected and titrated as for ammoniacal nitrogen.<sup>7</sup> Total nitrogen was determined by Kjeldahl salicylic acid reduction method to include nitrate nitrogen (Jackson, 1958)<sup>4</sup>.

### Results and Discussion

The results recorded in tables I and II represent the means of  $\pm$  two replicates of each treatment.

Addition of gamma BHC and telodrin to the soil apparently increases its available nitrogen content and the effect is more pronounced with increased

moisture level. Bollen *et al*<sup>2</sup> have also reported beneficial action of gamma BHC on bacterial population and their physiological activity. In a laboratory study Jaiswal<sup>5</sup> observed that application of telodrin and gamma BHC in moderate dosage increases bacterial population and nitrification of ammonium sulphate. The increase in available nitrogen content appears to be as a result of intervention of traces of insecticides in enzymes or other catalytic system involved in the nutrition of the micro-organisms responsible for mineralisation of soil organic matter. More pronounced increases in available nitrogen under higher moisture level may be on account of greater activity of these insecticides at higher moisture level. It is in line with the observations of Harris<sup>3</sup> who reported greater activity of a number of insecticides with increasing moisture level.

With higher dose of telodrin application (25 ppm.) there was consistent accumulation of ammoniacal nitrogen under both the moisture regimes. This is probably due to suppression of the activity of nitrosomonas the bacteria responsible for conversion of ammonium ion into nitrite ion. Srivastava<sup>9</sup> has also observed suppression of nitrification of ammonium sulphate by the application of telodrin.

The results recorded in tables I and II show that when sodium nitrate is added to the soil there is a gradual loss of nitrogen and it is more pronounced with higher moisture level. Presence of excess of moisture in the soil is very likely to exclude air from soil interspaces and create anaerobic conditions and thus accentuate the process of denitrification and increase the loss of nitrogen. It can also be seen that in the treatments where the insecticides have been added especially in higher doses, the total available nitrogen (nitrate + ammoniacal) is invariably more. This could happen either by increased availability of soil interlocked nitrogen or decreased loss of fertilizer nitrogen or both. Higher total nitrogen values observed under insecticidal treatments shows that insecticides reduce the loss of nitrogen. Increased nitrogen fixation observed by Jaiswal<sup>5</sup> with the addition of gamma BHC and telodrin can also be explained on the basis of this contention.

From the above findings it can be concluded that application of gamma BHC and telodrin to the soil liberates available nitrogen interlocked in soil organic matter. Telodrin at higher rate of application favours accumulation of ammoniacal nitrogen. The insecticides also seem to retard the process of denitrification and reduce loss of fertilizer nitrogen. The fertilizer conserving action of the insecticides, thus seems to be partly responsible for boosting the cane yield.

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## On Dual Integral Equations

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### Abstract

In this paper a formal solution of the pair of dual integral equations :

$$\int_0^{\infty} \alpha^{1+2\mu-2\nu} [1+R(\alpha)] \phi(\alpha) (\alpha x)^{\nu+1} J_{\nu}(\alpha x) d\alpha = f(x), 0 < x < 1$$

$$\int_0^{\infty} \phi(\alpha) (\alpha x)^{\mu+1} J_{\mu}(\alpha x) d\alpha = 0, x > 1, \mu > -\frac{3}{2}, -\frac{1}{2} < \mu - \nu < \frac{1}{2},$$

where  $R(\alpha)$  and  $f(x)$  are prescribed functions of  $\alpha$  and  $x$ , and  $\phi(\alpha)$  is unknown is derived (in the sense of reduction to a Fredholm integral equation of second type which can be best solved numerically).

### 1. Introduction

Dual integral equations of some complicated type arise in the discussion of mixed boundary value problems. They can be written in the form

$$\int_0^{\infty} \alpha^{-p} [1+R(\alpha)] \phi(\alpha) J_{\nu}(\alpha x) d\alpha = f(x), 0 < x < 1 \quad (1.1)$$

$$\int_0^{\infty} \alpha^{-q} \phi(\alpha) J_{\mu}(\alpha x) d\alpha = F(x), x > 1 \quad (1.2)$$

where the functions  $R(\alpha)$ ,  $f(x)$  and  $F(x)$  are prescribed in the ranges stated and  $\phi(\alpha)$  is to be determined.

A solution of these equations in the case  $\nu = \mu = 0$ , and  $F(x) \equiv 0$  was given by Tranter (7) and for the case  $\mu = \nu$  was given by him (8) in 1954, Cooke (1) gave a solution which is the integral analogue of Tranter's method; in it the unknown function satisfies a linear integral equation of the second kind of Fredholm's type. A special case of physical interest was discussed by the same method derived independently by Lebedev and Ufliand (4). A general solution, derived by a different method, was also given by Noble (5). In all these investigation it was assumed that  $F(x) \equiv 0$ .

The object of the present paper is to obtain a solution of the pair of dual integral equations :

$$\int_0^{\infty} \alpha^{1+2\mu-2\nu} [1+R(\alpha)] \phi(\alpha) (\alpha x)^{\nu+1} J_{\nu}(\alpha x) d\alpha = f(x), 0 < x < 1 \quad (1.3)$$

$$\int_0^{\infty} \phi(\alpha) (\alpha x)^{\mu+1} J_{\mu}(\alpha x) d\alpha = 0, x > 1 \quad \mu > -3/2, -\frac{1}{2} < \mu - \nu < \frac{1}{2}, \quad (1.4)$$

where  $R(\alpha)$  and  $f(x)$  are prescribed functions of  $\alpha$  and  $x$  and  $\phi(\alpha)$  is to be determined, by a generalization of a method due to Srivastava (6) who obtained a solution for the special case when  $\mu = \nu$ . The analysis given here is purely formal.

## 2. Solution of dual integral equations (1.3) and (1.4)

We set

$$\phi(\alpha) = \int_0^1 (\alpha t)^{-\mu-\frac{1}{2}} \psi(t) J_{\mu+\frac{1}{2}}(\alpha t) dt \quad (2.1)$$

where  $\psi(t)$  is a function to be determined. The equation (1.4) is automatically satisfied with the value of  $\phi(\alpha)$  given by (2.1) by virtue of the result [3, p. 48(7)]

$$\int_0^\infty J_\lambda(at) J_\mu(bt) t^{1+\mu-\lambda} dt = \begin{cases} 0 & , 0 < a < b \\ \frac{b^\mu (a^2 - b^2)^{\lambda-\mu-1}}{2^{\lambda-\mu-1} a^\lambda \Gamma(\lambda-\mu)} & , 0 < b < a \end{cases} \quad (2.2)$$

$$Re(\lambda) > Re(\mu) > -1.$$

To deal with the equation (1.3), we integrate it w. r. t.  $x$  from 0 to  $x$  and thus we get

$$G(x) = \int_0^x f(x) dx = \int_0^\infty \alpha^{2\mu-2\nu} [1+R(\alpha)] \phi(\alpha) (\alpha x)^{\nu+1} J_{\nu+1}(\alpha x) d\alpha \quad (2.3)$$

Now substituting the value of  $\phi(\alpha)$  from (2.1) in (2.3) we get

$$\begin{aligned} & \int_0^\infty \int_0^1 \alpha^{\mu-\nu-\frac{1}{2}} \psi(t) J_{\mu+\frac{1}{2}}(\alpha t) J_{\nu+1}(\alpha x) t^{-\mu-\frac{1}{2}} x^{\nu+1} d\alpha dt \\ & + \int_0^\infty \alpha^{\mu-\nu+\frac{1}{2}} R(\alpha) d\alpha \int_0^1 t^{-\mu-\frac{1}{2}} x^{\nu+1} J_{\mu+\frac{1}{2}}(\alpha t) J_{\nu+1}(\alpha x) dt = G(x), x < 1 \end{aligned} \quad (2.4)$$

Now the first integral in (2.4) is equivalent to

$$\begin{aligned} & \int_0^1 x^{\nu+1} t^{-\mu-\frac{1}{2}} \psi(t) dt \int_0^\infty \alpha^{\mu-\nu+\frac{1}{2}} J_{\mu+\frac{1}{2}}(\alpha t) J_{\nu+1}(\alpha x) d\alpha \\ & = \frac{2^{\mu-\nu+\frac{1}{2}}}{\Gamma(\nu-\mu+\frac{1}{2})} \int_0^x \frac{\psi(t) dt}{(x^2-t^2)^{\mu-\nu+\frac{1}{2}}}, \end{aligned} \quad (2.5)$$

$$x < 1, (\nu+1) > (\mu+\frac{1}{2}) > -1.$$

by virtue of the result [3, p. 47(8)]

$$\int_0^\infty t^{1+\lambda-\mu} J_\lambda(at) J_\mu(bt) dt = \begin{cases} 0 & , 0 < b < a \\ \frac{2^{\lambda-\mu+1} a^\lambda}{\Gamma(\mu-\lambda) b^\mu} (b^2-a^2)^{\mu-\lambda-1}, 0 < a < b \end{cases} \quad (2.6)$$

$$Re(\mu) > Re(\lambda) > -1$$

The second integral of (2.4) can be written as

$$\int_0^1 \left(\frac{x}{t}\right)^{\mu+\frac{1}{2}} \psi(t) dt \int_0^\infty (\alpha x)^{\nu-\mu+\frac{1}{2}} J_{\mu+\frac{1}{2}}(\alpha t) J_{\nu+1}(\alpha x) R(\alpha) d\alpha, x < 1. \quad (2.7)$$

By virtue of the result,

$$J_{\mu+\frac{1}{2}}(\alpha t) = \frac{2^{\frac{1}{2}+v-\mu} a^{\mu-v+\frac{1}{2}}}{t^{\mu+\frac{1}{2}} \Gamma(\mu-v+\frac{1}{2})} \int_0^t \frac{u^{v+1} J_v(\alpha u) du}{(t^2-u^2)^{v-\mu+\frac{1}{2}}} \quad (2.8)$$

the second integral of (2.7) is equivalent to

$$\begin{aligned} & \int_0^t \frac{t^{v-\mu} (u/t)^{v+\frac{1}{2}}}{(t^2-u^2)^{v-\mu+\frac{1}{2}}} \left[ \int_0^\infty (\alpha u)^{\frac{1}{2}} x^{v-\mu} (\alpha x)^{\frac{1}{2}} J_v(\alpha u) J_{v+1}(\alpha x) R(\alpha) d\alpha \right] du \\ &= \frac{2}{\pi} \int_0^t \frac{t^{v-\mu} (x/t)^{v+\frac{1}{2}} H(u, x)}{(t^2-u^2)^{v-\mu+\frac{1}{2}}} du \end{aligned} \quad (2.9)$$

where

$$H(u, x) = \frac{\pi 2^{v-\mu+\frac{3}{2}}}{\Gamma(v-\mu+\frac{1}{2})} (u/x)^{v+\frac{1}{2}} x^{v-\mu} \int_0^\infty R(\alpha) (\alpha x)^{\frac{1}{2}} J_{v+1}(\alpha x) (\alpha u)^{\frac{1}{2}} J_v(\alpha u) d\alpha \quad (2.10)$$

Hence the result (2.7) can be replaced by

$$\frac{2}{\pi} \int_0^1 t^{v-\mu} (x/t)^{v+\mu+1} \psi(t) dt \int_0^t \frac{H(u, x) du}{(t^2-u^2)^{v-\mu+\frac{1}{2}}} \quad (2.11)$$

Substituting the values of first and second integral from (2.5) and (2.11) in the result (2.4), we get

$$\begin{aligned} & \frac{2^{\mu-v+\frac{1}{2}}}{\Gamma(v-\mu+\frac{1}{2})} \int_0^x \frac{\psi(t) dt}{(x^2-t^2)^{\mu-v+\frac{1}{2}}} + \frac{2}{\pi} \int_0^1 t^{v-\mu} \left(\frac{x}{t}\right)^{v+\mu+1} \psi(t) dt \int_0^t \frac{H(u, x) du}{(t^2-u^2)^{v-\mu+\frac{1}{2}}} \\ &= G(x), \quad x < 1 \end{aligned} \quad (2.12)$$

If  $h(x)$  and  $h'(x)$  be continuous in  $0 \leq x \leq a$  and  $0 < k < 1$  then the solution of

$$\int_0^x \psi(t) (x^2-t^2)^{-k} dt = h(x) \quad (2.13)$$

is

$$\psi(t) = \frac{2 \sin \pi k}{\pi} \frac{d}{dt} \int_0^t v h(v) (t^2-v^2)^{k-1} dv \quad (2.14)$$

This is a simple transformation of Abel's integral equation as given by Copson (2).

Using (2.13) and (2.14) in (2.12), we obtain

$$\begin{aligned} & \frac{2^{\mu-v+\frac{1}{2}}}{\Gamma(v-\mu+\frac{1}{2})} h(x) + \frac{4}{\pi^2} \sin \pi(\mu-v+\frac{1}{2}) \int_0^1 M(x, t) \times \\ & \left( \frac{d}{dt} \int_0^t v h(v) (t^2-v^2)^{\mu-v-\frac{1}{2}} dv \right) dt = G(x), \quad x < 1, \quad 0 < (\mu-v+\frac{1}{2}) < 1 \end{aligned} \quad (2.15)$$

where

$$M(x, t) = t^{v-\mu} \left(\frac{x}{t}\right)^{v+\mu+1} \int_0^t \frac{H(u, x) du}{(t^2-u^2)^{v-\mu+\frac{1}{2}}} \quad (2.16)$$

Now integrating by parts, we obtain

$$\begin{aligned}
 & \int_0^1 M(x, t) \left( \frac{d}{dt} \right) \int_0^t v h(v) (t^2 - v^2)^{\mu-v-\frac{1}{2}} dv dt \\
 &= \left[ M(x, t) \int_0^t v h(v) (t^2 - v^2)^{\mu-v-\frac{1}{2}} dv \right]_0^1 - \int_0^1 M_t^1(x, t) \int_0^t \frac{v h(v) dv}{(t^2 - v^2)^{\frac{1}{2} + v - \mu}} dt \\
 &= M(x, 1) \int_0^1 \frac{v h(v) dv}{(1 - v^2)^{\frac{1}{2} + v - \mu}} - \int_0^1 v h(v) dv \int_0^1 \frac{M_t^1(x, t) dt}{(t^2 - v^2)^{\frac{1}{2} + v - \mu}} \\
 &= \int_0^1 h(v) k(x, v) dv, \tag{2.17}
 \end{aligned}$$

where

$$k(x, v) = \frac{v M(x, 1)}{(1 - v^2)^{\frac{1}{2} + v - \mu}} - v \int_0^1 \frac{M_t^1(x, t) dt}{(t^2 - v^2)^{\frac{1}{2} + v - \mu}} \tag{2.18}$$

Now using the result (2.17) in (2.15), we get

$$\frac{2^{\mu-v+\frac{1}{2}}}{\Gamma(\frac{1}{2} - \mu + v)} h(x) + \frac{4}{\pi^2} \sin \pi(\frac{1}{2} + \mu - v) \int_0^1 h(v) k(x, v) dv = G(x), \quad x < 1 \tag{2.19}$$

This is a Fredholm integral equation of second kind, which can be best solved numerically. Knowing  $h(x)$ , we can obtain  $\psi(t)$  from (2.14) and thus  $\phi(\alpha)$  can be calculated from (2.1).

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## Methods and Materials

Canal Water was used for irrigation of the crops.

*Rate of addition of the amendments per acre :*

The experimental results are recorded in the subsequent pages.

In the data, the following abbreviations have been used for the phosphates :

G.B.S. for German basic slag

T.B.S. for Tata basic slag

TABLE 1

*Chemical composition (%) of materials (oven dried)*

	Soil + grasses	Paddy straw	Wheat straw	Soil	German basic slag	Tata basic slag
Loss on Ignition	90.4263	90.0750	—	3.9153	—	—
Ash	9.5737	9.6251	8.985	96.0847	—	—
HCl insoluble	5.0688	5.4712	—	81.6130	—	—
Sesquioxide	1.9732	1.9007	—	11.2071	—	—
Fe <sub>2</sub> O <sub>3</sub>	0.8716	0.9251	0.501	4.2630	7.2000	7.132
CaO	0.8325	0.7053	0.745	0.9863	33.5500	25.324
P <sub>2</sub> O <sub>5</sub>	0.4727	0.3987	0.281	0.0815	17.9050	7.6013
K <sub>2</sub> O	0.7126	0.6641	1.038	0.9578	9.5250	10.12
MgO	0.2605	0.1892	0.197	0.5990	5.5050	3.969
Available P <sub>2</sub> O <sub>5</sub>	—	—	—	0.0164	9.3969	3.825
Total-C	35.7274	34.3135	38.462	0.5648	—	—
Total-N	0.6001	0.5368	0.624	0.0553	—	—
NH <sub>3</sub> -N	—	—	—	0.0041	—	—
NO <sub>3</sub> -N	—	—	—	0.0061	—	—
C/N	59.5	64.1	61.6	10.2	—	—

TABLE 2

*Composition of soil (%) before Paddy transplantation*

TREATMENTS						
	Soil alone (control)	Soil + Wheat straw	Soil + Wheat straw + T.B.S.	Soil + Wheat straw + G.B.S.	Soil + grasses	Soil + grasses + T.B.S.
CaO	0.9861	1.0468	1.4115	1.4160	1.0517	1.4131
P <sub>2</sub> O <sub>5</sub>	0.0815	0.0886	0.1172	0.1187	0.0880	0.1169
Available P <sub>2</sub> O <sub>5</sub>	0.0169	0.0238	0.0485	0.0498	0.0231	0.0473
K <sub>2</sub> O	0.9576	1.0280	1.1291	1.1323	1.0310	1.1303
MgO	0.5990	0.6194	0.6536	0.6587	0.6297	0.6631
Total-C	0.5619	0.7415	0.6841	0.6821	0.7321	0.6744
Total-N	0.0550	0.0754	0.0911	0.0920	0.0748	0.0898
NH <sub>3</sub> -N	0.0040	0.0059	0.0085	0.0088	0.0058	0.0083
NO <sub>3</sub> -N	0.0061	0.0082	0.0118	0.0122	0.0079	0.0115
C/N	10.2	9.8	7.3	7.2	9.6	7.3

Table 2—(contd.)

	Soil+grasses +G.B.S.	Soil+Paddy straw	Soil+Paddy straw+T.B.S.	Soil+Paddy straw+G.B.S.	Soil+ T.B.S.	Soil+ G.B.S.
CaO	1.4178	1.0442	1.4052	1.4093	1.3062	1.3126
P <sub>2</sub> O <sub>5</sub>	0.1181	0.0869	0.1158	0.1167	0.1114	0.1135
Available P <sub>2</sub> O <sub>5</sub>	0.0485	0.0220	0.0450	0.0465	0.0231	0.0240
K <sub>2</sub> O	1.1334	1.0131	0.1227	1.1260	1.0811	1.0836
MgO	0.6669	0.6240	0.6578	0.6619	0.6343	0.6381
Total-C	0.6708	0.7236	0.6689	0.6657	0.5592	0.5587
Total-N	0.0909	0.0737	0.0873	0.0881	0.0551	0.0552
NH <sub>3</sub> -N	0.0085	0.0056	0.0080	0.0081	0.0040	0.0041
NO <sub>3</sub> -N	0.0119	0.0077	0.0109	0.0113	0.0062	0.0062
C/N	7.2	9.5	7.4	7.3	10.1	10.1

TABLE 3

*Composition of Soil (%) after harvesting of Paddy crop*

## TREATMENTS

	Soil alone (control)	Soil+Wheat straw	Soil+ Wheat straw+ T.B.S.	Soil+ Wheat straw+ G.B.S.	Soil+ grasses	Soil+ grasses+ T.B.S.
CaO	0.9782	1.0156	1.3402	1.3452	1.0172	1.3419
P <sub>2</sub> O <sub>5</sub>	0.0809	0.0858	0.1120	0.1138	0.0849	0.1115
Available P <sub>2</sub> O <sub>5</sub>	0.0161	0.0271	0.0531	0.0548	0.0259	0.0518
K <sub>2</sub> O	0.9501	1.0001	1.0552	1.0578	1.0019	1.0561
MgO	0.5980	0.6112	0.6351	0.6370	0.6179	0.6411
Total-C	0.5581	0.6979	0.6321	0.6288	0.6887	0.6236
Total-N	0.0544	0.0722	0.0861	0.0867	0.0717	0.0849
NH <sub>3</sub> -N	0.0038	0.0055	0.0079	0.0082	0.0053	0.0075
NO <sub>3</sub> -N	0.0059	0.0077	0.0110	0.0113	0.0074	0.0109
C/N	10.2	9.6	7.3	7.2	9.6	7.3

	Soil+ grasses+ G.B.S.	Soil+ Paddy straw	Soil+ Paddy straw+ T.B.S.	Soil+ Paddy straw+ G.B.S.	Soil+ T.B.S.	Soil+ G.B.S.
CaO	1.3467	1.0122	1.3233	1.3392	1.2807	1.2846
P <sub>2</sub> O <sub>5</sub>	0.1129	0.0837	0.1102	0.1112	0.1082	0.1094
Available P <sub>2</sub> O <sub>5</sub>	0.0532	0.0248	0.0500	0.0518	0.0240	0.0249
K <sub>2</sub> O	1.0585	0.9854	1.0464	1.0509	1.0541	1.0559
MgO	0.6432	0.6131	0.6392	0.6409	0.6241	0.6273
Total-C	0.6201	0.6724	0.6177	0.6168	0.5551	0.5546
Total-N	0.0858	0.0702	0.0833	0.0841	0.0545	0.0546
NH <sub>3</sub> -N	0.0076	0.0052	0.0073	0.0073	0.0039	0.0039
NO <sub>3</sub> -N	0.0113	0.0072	0.0101	0.0106	0.0060	0.0061
C/N	7.2	9.5	7.4	7.3	10.1	10.1

**TABLE 4**  
*Composition of Soil (%) after harvesting of Wheat crop*

	TREATMENTS					
	Soil alone (control)	Soil + Wheat straw	Soil + Wheat straw + T.B.S.	Soil + Wheat straw + G.B.S.	Soil + grasses	Soil + grasses + T.B.S.
CaO	0.9721	0.9946	1.2567	1.2605	0.9941	1.2559
P <sub>2</sub> O <sub>5</sub>	0.0805	0.0837	0.1065	0.1071	0.0827	0.1048
Available P <sub>2</sub> O <sub>5</sub>	0.0146	0.0251	0.0502	0.0518	0.0241	0.0487
K <sub>2</sub> O	0.9439	0.9789	1.0048	1.0078	0.9774	1.0043
MgO	0.5972	0.6025	0.6212	0.6235	0.6080	0.6345
Total-C	0.5473	0.6616	0.6002	0.5958	0.6522	0.5926
Total-N	0.0537	0.0671	0.0791	0.0794	0.0665	0.0779
NH <sub>3</sub> -N	0.0037	0.0050	0.0069	0.0071	0.0048	0.0066
NO <sub>3</sub> -N	0.0056	0.0071	0.0100	0.0103	0.0067	0.0096
C/N	10.1	9.8	7.5	7.5	9.8	7.6

	Soil + grasses + T.B.S.	Soil + Paddy straw	Soil + Paddy straw + T.B.S.	Soil + Paddy straw + G.B.S.	Soil + T.B.S.	Soil + G.B.S.
CaO	1.2598	0.9912	1.2390	1.2462	1.2715	1.2727
P <sub>2</sub> O <sub>5</sub>	0.1054	0.0814	0.1037	0.1042	0.1054	0.1065
Available P <sub>2</sub> O <sub>5</sub>	0.0499	0.0220	0.0469	0.0482	0.0212	0.0229
K <sub>2</sub> O	1.0062	0.9663	0.9975	0.9993	0.0336	0.0351
MgO	0.6362	0.6043	0.6283	0.6299	0.6205	0.6232
Total-C	0.5890	0.6466	0.5856	0.5851	0.5427	0.5414
Total-N	0.0784	0.0658	0.0763	0.0768	0.0539	0.0539
NH <sub>3</sub> -N	0.0068	0.0047	0.0062	0.0063	0.0036	0.0037
NO <sub>3</sub> -N	0.0100	0.0064	0.0094	0.0098	0.0058	0.0058
C/N	7.5	9.8	7.6	7.6	10.0	10.0

**TABLE 5**  
*Yield of Paddy grain in Kgms.*

Treatments	BLOCKS						Total
	1	2	3	4	5	6	
Soil + alone (control)	10.50	9.00	11.25	9.25	11.50	12.00	63.50
Soil + Wheat straw	18.00	18.50	16.50	17.25	16.75	18.75	105.75
Soil + Wheat straw + T.B.S.	21.50	23.50	22.50	22.75	24.25	23.00	137.50
Soil + Wheat straw + G.B.S.	24.75	23.25	23.50	26.26	25.25	25.75	148.75



Table 5—(c *ntd.*)

Soil + grasses	17.00	18.50	17.25	16.25	18.25	16.50	103.75
Soil + grasses + T.B.S	22.00	23.50	24.00	20.50	23.00	22.25	135.25
Soil + grasses + G.B.S.	24.50	25.50	24.00	22.50	25.00	23.00	144.50
Soil + Paddy straw	16.25	17.00	16.50	15.75	17.25	15.25	98.00
Soil + Paddy straw+T.B.S.	20.75	21.00	19.75	22.00	18.75	19.50	121.75
Soil + Paddy straw+G.B.S	20.50	23.00	21.50	21.25	22.50	22.25	131.00
Soil + T.B.S.	11.00	11.50	12.75	14.00	12.00	11.25	72.50
Soil + G.B.S.	12.25	11.00	11.25	13.50	12.50	13.75	74.25
Total	219.00	225.25	220.75	221.25	227.00	223.25	1336.50

*Analysis of Variance of Yield data of Paddy grain*

Sources of variation	D.F.	S.S.	M.S.	Calculated F	F from Table 1%	Table 5%	Leave of Significance
Blocks	5	3.74	0.748	0.913	3.445	2.385	
Treatments	11	1664.94	151.35	184.7	5.32	3.20	Very highly significant
Error	55	45.09	0.819	—	—	—	—
Total	71	1713.77	—	—	—	—	—

TABLE 6  
*Yield of Paddy straw in Kgms.*

Treatments	BLOCKS						Total
	1	2	3	4	5	6	
Soil alone (control)	27	29	28	30	25	26	165
Soil + Wheat straw	49	51	53	48	47	50	298
Soil+Wheat straw + T.B.S.	61	65	63	65	64	67	385
Soil+Wheat straw + G.B.S.	67	65	66	69	63	69	399
Soil+grasses	49	50	53	47	49	48	296
Soil + grasses + T.B.S.	60	67	63	60	62	65	377
Soil + grasses + G.B.S.	65	64	60	68	63	69	389
Soil + Paddy straw	46	44	46	44	42	43	265
Soil+Paddy straw + T.B.S.	62	60	62	61	64	57	366
Soil+ Paddy straw + G.B.S.	61	64	61	68	63	61	378
Soil + T.B.S.	29	33	29	35	31	36	193
Soil + G.B.S.	34	30	36	31	35	33	199
Total	610	622	620	626	608	624	3710

Table 6—(contd.)

## Analysis of Variance of Yield data of Paddy straw

Sources of Variation	D.F.	S.S.	M.S.	Calculate F	F from table 1%	5%	Level of Significance
Blocks	5	23.66	4.732	0.17	3.445	2.385	Yield is significant
Treatments	11	13571.33	1233.75	44.33	5.32	3.20	
Error	55	1531.01	27.83	—	—	—	
Total	71	15126.00	—	—	—	—	

TABLE 7  
Yield of Wheat grain in Kgms.

Treatments	BLOCKS						Total
	1	2	3	4	5	6	
Soil + alone (control)	8.50	9.50	9.00	7.00	8.00	10.75	52.75
Soil + Wheat straw	14.25	14.75	12.50	13.25	16.00	13.50	84.25
Soil + Wheat straw + T.B.S.	17.00	19.25	21.50	20.25	20.25	19.50	117.75
Soil + Wheat straw + G.B.S.	20.50	19.25	20.25	22.00	20.75	23.50	126.25
Soil + grasses	13.00	14.25	12.00	13.00	12.50	14.50	79.25
Soil + grasses + T.B.S.	17.50	18.00	20.25	19.00	20.75	17.00	112.50
Soil + grasses + G.B.S.	21.00	18.50	19.25	21.00	21.50	18.50	119.75
Soil + Paddy straw	13.25	11.00	12.00	14.00	12.00	11.75	74.00
Soil + Paddy straw + T.B.S.	15.25	17.50	18.75	16.25	18.25	16.50	102.50
Soil + Paddy straw + G.B.S.	19.25	18.50	17.00	20.25	17.00	18.00	110.00
Soil + T.B.S.	9.50	9.75	8.00	10.50	11.00	11.25	60.00
Soil + G.B.S.	10.00	9.50	11.50	10.25	9.00	13.00	63.25
Total	179.00	179.75	182.00	186.75	187.00	187.75	1102.25

## Analysis of Variance of Yield data of Wheat grain

Sources of Variation	D.F.	S.S.	M.S.	Calculated F	F from table 1%	5%	Level of Significance
Blocks	5	6.82	1.164	0.657	3.445	2.385	Yield is highly significant.
Treatments	11	1233.02	112.09	63.32	5.32	3.20	
Error	55	97.77	1.77	—	—	—	
Total	71	1337.61	—	—	—	—	

TABLE 8  
*Yield of Wheat straw in Kgms.*

Treatments	B L O C K S						Total
	1	2	3	4	5	6	
Soil alone (control)	21	22	19	18	20	21	121
Soil + Wheat straw	32	32	34	33	36	35	202
Soil + Wheat straw + T. B. S.	46	47	48	45	43	45	274
Soil + Wheat straw + G. B. S.	53	50	49	50	48	47	297
Soil + grasses	33	29	34	34	31	30	191
Soil + grasses + T. B. S.	42	46	40	45	43	42	258
Soil + grasses + G. B. S.	43	48	43	46	45	49	274
Soil + Paddy straw	28	29	33	29	32	30	181
Soil + Paddy straw + T. B. S.	41	40	42	39	38	43	243
Soil + Paddy straw + G. B. S.	47	45	46	45	42	43	268
Soil + T. B. S.	24	25	25	21	22	23	140
Soil + G. B. S.	25	23	24	26	24	27	149
Total	435	436	437	431	424	435	2598

*Analysis of Variance of Yield data of Wheat straw*

Sources of Variation	D. F	S. S	M. S	Calculated F	F from table 1%	F from table 5%	Level of significance
Blocks	5	10.11	2.22	1.1	3.445	2.385	Very highly significant
Treatments	11	6636.78	603.34	311.00	5.32	3.20	
Error	56	106.89	1.99	—	—	—	
Total	71	6753.78	—	—	—	—	

TABLE 9  
*Yield of Paddy grain in Kgms.*

Treatments	B L O C K S						Total
	1	2	3	4	5	6	
Soil alone (control)	11.00	9.25	10.25	8.50	11.75	10.50	61.25
Soil + Wheat straw	15.50	13.50	14.75	13.75	12.75	12.50	82.75
Soil + Wheat straw + T.B.S.	15.75	18.25	18.75	16.50	16.00	17.50	102.75
Soil + Wheat straw + G.B.S.	16.75	17.75	18.00	17.00	16.25	19.50	105.25
Soil + grasses	13.50	13.00	15.00	12.50	12.25	12.75	79.00
Soil + grasses + T.B.S.	15.75	14.50	17.75	16.50	16.00	16.75	97.25
Soil + grasses + G.B.S.	15.00	16.50	16.75	18.00	15.50	17.75	99.50

Table 9—(contd.)

Soil + Paddy straw	13.50	12.00	11.50	15.00	13.75	12.50	78.25
Soil + Paddy straw + T.B.S.	14.50	17.25	17.50	14.75	15.75	14.50	94.25
Soil + Paddy straw + G.B.S.	17.00	14.50	15.25	17.50	17.00	15.75	97.00
Soil + T.B.S.	10.25	13.00	11.25	10.00	12.50	10.75	67.75
Soil + G.B.S.	10.25	11.50	11.00	12.00	10.75	13.00	68.50
Total	168.75	171.00	177.75	172.00	170.25	173.75	1033.50

## Analysis of Variance of Yield data of Paddy grain

Sources of Variation	D. F.	S. S.	M. S.	Calculated F	F from table		Level of significance
					1%	5%	
Blocks	5	4.198	0.839	0.527	3.445	2.385	Highly significant
Treatments	11	420.052	38.186	24.01	5.32	3.20	
Error	55	87.511	1.59	—	—	—	
Total	71	511.761	—	—	—	—	

TABLE 10  
Yield of Paddy straw in Kgms.

Treatments	B L O C K S						Total
	1	2	3	4	5	6	
Soil alone (control)	22	25	27	26	26	25	151
Soil + Wheat straw	35	31	33	34	37	37	207
Soil + Wheat straw + T. B. S.	36	42	40	42	39	41	240
Soil + Wheat straw + G. B. S.	40	42	45	39	44	41	251
Soil + grasses	32	36	33	35	30	34	200
Soil + grasses + T. B. S.	37	39	36	41	35	40	228
Soil + grasses + G. B. S.	42	39	38	36	41	42	238
Soil + Paddy straw	33	30	32	27	32	29	183
Soil + Paddy straw + T. B. S.	35	33	39	37	34	38	216
Soil + Paddy straw + G. B. S.	40	41	38	35	39	38	231
Soil + T. B. S.	27	29	28	29	25	26	164
Soil + G. B. S.	23	29	31	29	26	25	168
Total	407	416	420	410	408	416	2474

*Analysis of Variance of Yield data of Paddy straw*

Sources of Variation	D. F.	S. S.	M. S.	Calculated F	F from table 1%	F from table 5%	Level of significance
Blocks	5	10.01	2.00	0.55	3.445	2.385	Highly significant
Treatments	11	2008.02	182.54	50.28	5.32	3.20	
Error	55	199.90	3.634	-	-	-	
Total	71	2217.94	-	-	-	-	

### Discussion

From an examination of the foregoing results it is observed that by applying different organic materials like Wheat straw, mixture of grasses and paddy straw to the field, higher yield both of wheat and paddy were obtained than in the 'Control' plots in which no organic materials have been added. It is further observed that mixtures of phosphates and organic matter is better to increase grain yields of wheat and paddy than organic matter alone. The efficiency of different treatments in increasing the yield of paddy and Wheat crops was in the following order :

Phosphated wheat straw > Phosphated grasses > Phosphated paddy straw  
> Wheat straw > Grasses > Paddy straw > G. B. S. > T. B. S. > Control.

The analysis of variances of the yield data of paddy grain and paddy straw (vide table-5 and table-6) show that the calculated values of F for grain and straw are 184.7 and 44.33 respectively which are highly significant at 10 % and 5 % levels.

The analysis of variances of the yield data of wheat grain and wheat straw, according to table-7 and table-8 respectively also show highly significant effects of the treatments, as well as amongst principal variants.

The yield of paddy grain and paddy straw (3rd crop) are recorded in tables 9 and 10. Analysis of variance of the yield data shows highly significant effects due to treatments. The 3rd crop of paddy showed lower yields of grain and straw in comparison to the first paddy crop. This is due to the fact that the earlier crop of paddy and wheat might have taken up a part of the plant nutrients as is evident from the analysis of the soil before transplantation of paddy (vide table-2) after harvesting of wheat crop (vide table-4).

It is very interesting to note that in these field trials there is a greater increase in the yield of paddy and wheat crops when organic materials are used mixed with phosphates than when used alone. The difference in the yield of crops is due to the presence of higher amounts of total nitrogen, available nitrogen and available phosphate as is evident from the analysis of the soil after decomposition of organic materials and before transplantation of the paddy crop (vide table-2). The plots treated with organic materials and phosphate show greater amounts of total nitrogen, available nitrogen and available phosphate than plots which had no phosphate. Results recorded in tables 3 and 4 show that the percentage of total nitrogen in soil is greater in phosphate treated plots.

Karraker, working in the U.S.A., obtained the following results showing marked nitrogen fixation by mixture of manure and phosphates :

Average of the three field treatments	Nitrogen in soil lbs/acre	Corn yield Bushels/acre
No manure	1600	17
Manure	1760	36
Manure + phosphate	1990	51

According to Dhar and co workers<sup>1</sup>, a mixture of organic substances like Farmyard manure, Straw, plant residues etc. and phosphates, when incorporated in soil, can build up the soil fertility permanently by fixing atmospheric nitrogen and supplying available nitrogen, phosphate, potash, trace elements and humus and thus maintain soil fertility.

Lady Eve Balfour<sup>2</sup> of The Soil Association, New Bells Farm, Haughley, Suffolk, England, reported that the plot to which basic slag was added to the straw contained the largest amount of nitrogen and produced the biggest crop 30.4 cwt. barley grain per acre, whilst the land to which 112 lbs. N was added, yielded only 20.8 cwt. barley grain per acre. The control plot produced 14 cwt of barley grains. Similar results were obtained at Burdwan, Midnapore and Birbhum (Bengal) in India<sup>3</sup> by the incorporation of straw and phosphates together.

It can, therefore, be concluded from the experimental results that a mixture of organic matter and phosphates is highly beneficial for increasing land fertility and production of crops.

#### References

1. Dhar, N. R. Presidential Address to 48th Indian Science Congress, p. 11, 1961.
2. Compare Dhar, N. R. Annual Progress Report on the Scheme of photo-chemical nitrogen Fixation in soils for 1956-60. p. 6, (Sheila Dhar Institute of Soil Science, Allahabad University).
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## On the flow of gas in the Barred Galaxy taking into account the resistance of the medium

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### Abstract

Motion of gas element has been investigated for the uniformly rotating, homogeneous cylinder, model of the barred galaxy. We have taken into account the resistance, of the medium, encountered by the flow of gas, and the forces due to gravitation and rotation. We have considered resistance to vary as  $v^3$ ,  $v^5$  and  $v^p \cos p^{-1} \psi$  (where  $p$  is a positive even or odd integer, and  $\psi$  is the angle which the tangent makes with the abscissa). For all the cases, the cartesian coordinates,  $x$  and  $y$ , have been obtained as a function of the parameter ' $t$ '. These investigations are found to be readily applicable if we consider, an uniformly rotating, homogeneous elliptic cylinder model for the barred galaxy.

### Introduction

We have elsewhere discussed<sup>1,2</sup> the possibility of the outward motion of gas assuming the uniformly rotating, homogeneous cylinder and homogeneous elliptic cylinder model for the barred galaxy. The outward flow of gas was discussed under the influence of forces due to gravitation and rotation. Since the gas element is bound to experience some resistance, due to the presence of pervading gas and dust, we considered<sup>1</sup> it to vary as the velocity<sup>3</sup>. Since the variation of resistance, with velocity, is imperfectly understood even today we will assume different resistance laws, in this communication. The need of the authentic relation is keenly felt for a better understanding of the extra-galactic phenomenon.

#### Case 1 :

The basic equations, in rotating coordinates, considering forces to the gravitation and rotation, and assuming the resistance to vary as the cube of the velocity, can be written as

$$\left. \begin{aligned} \frac{d^2x}{dt^2} - 2\Omega \frac{dy}{dt} - \Omega^2 x &= X_c - Lv^3 \cos \psi \\ \frac{d^2y}{dt^2} + 2\Omega \frac{dx}{dt} - \Omega^2 y &= Y_c - Lv^3 \sin \psi \end{aligned} \right\} \quad (1)$$

where  $\Omega$  is the uniform angular velocity, of the barred galaxy.  $X_c$  and  $Y_c$  are the components of the gravitational force, at any internal point, of the cylinder, along the axes. The angle which the tangent makes with the abscissa is designated as  $\psi$ . It can be seen, as is usual on problems of galaxies, we have not considered effects due to pressure and the magnetic field.

In equation (1) substituting  $\dot{X}_c = -2 \pi G \rho x$ , and assuming the condition

$$V_c - L v^3 \sin \psi + \Omega^2 y = 0 \quad (2)$$

we have

$$\frac{d^2 y}{dt^2} = -2\Omega \frac{dx}{dt} \quad (3)$$

On integrating above equation, and applying the initial conditions,

$$\frac{dy}{dt} = -V_c \text{ when } x = x_0, \text{ we find}$$

$$\frac{dy}{dt} = V_c - 2 \Omega (x - x_0) \quad (4)$$

From equations (1) and (4), we have, on simplification

$$\frac{d^2 x}{dt^2} + (2 \pi G \rho + 3 \Omega^2) x = (2 \Omega V_c + 4 \Omega^2 x_0) - L \left( \frac{ds}{dt} \right)^2 \cdot \left( \frac{dx}{dt} \right) \quad (5)$$

As an approximation let us take<sup>1</sup> the values of  $\frac{dx}{dt}$  and  $\frac{dy}{dt}$ , when resistance is not considered, as follows :

$$\frac{dx}{dt} = a \sin bt \text{ and } \frac{dy}{dt} = c + d \cos bt \quad (6)$$

where

$$\left. \begin{aligned} a &= \frac{\Omega^2 x_0 + 2\Omega V_c - 2\pi G \rho x_0}{(2\pi G \rho + 3\Omega^2)^{1/2}} \\ b &= (2\pi G \rho + 3\Omega^2)^{1/2} \\ c &= \frac{(V_c + 2\Omega x_0)(2\pi G \rho - \Omega^2)}{2\pi G \rho + 3\Omega^2} \\ \text{and } d &= \frac{2\Omega(\Omega^2 x_0 + 2\Omega V_c - 2\pi G \rho x_0)}{2\pi G \rho + 3\Omega^2} \end{aligned} \right\} \quad (7)$$

$$\text{Now } \frac{dx}{dt} \cdot \left( \frac{ds}{dt} \right)^2 = a \sin bt \{ a^2 \sin^2 bt + (c + d \cos bt)^2 \} \quad (8)$$

which yields on simplification

$$\begin{aligned} \frac{dx}{dt} \cdot \left( \frac{ds}{dt} \right)^2 &= \left( \frac{3a^3}{4} + ac^2 + \frac{ad^2}{4} \right) \sin bt + acd \sin 2bt \\ &\quad + \left( -\frac{a^3}{4} + \frac{ad^2}{4} \right) \sin 3bt \end{aligned} \quad (9)$$

From equations (5), (7) and (9), we have

$$\begin{aligned} \frac{d^2 x}{dt^2} + b^2 x &= (2\Omega V_c + 4\Omega^2 x_0) - L \left( \frac{3a^3}{4} + ac^2 + \frac{ad^2}{4} \right) \sin bt \\ &\quad - L acd \sin 2bt - L \left( \frac{ad^2}{4} - \frac{a^3}{4} \right) \sin 3bt \end{aligned} \quad (10)$$



The solution of which is easily seen to be

$$x = C_1 \cos bt + C_2 \sin bt + \frac{2\Omega V_c + 4\Omega x_0}{b^2} + \frac{L}{2b} \left( \frac{3a^2}{4} + ac^2 + \frac{ad^2}{4} \right) \\ t \cos bt + \frac{Lacd}{3b^2} \sin 2bt + \frac{L}{8b^2} \left( \frac{ad^2}{4} - \frac{a^3}{4} \right) \sin 3bt \quad (11)$$

where  $C_1$  and  $C_2$  are arbitrary constants. Their values can be derived, using the initial conditions, as

$$\left. \begin{aligned} C_1 &= -\frac{a}{b_2} \\ \text{and } C_2 &= \frac{-La}{96b_2} (27a^2 + 48c^2 + 21d^2 + 64cd) \end{aligned} \right\} \quad (12)$$

From equations (4) and (11), we find, on simplification

$$\frac{dy}{dt} = \frac{(V_c + 2\Omega x_0)(2\pi G\rho - \Omega^2)}{b^2} - 2\Omega C_2 \sin bt - \frac{2\Omega Lacd}{3b^2} \sin 2bt \\ - \frac{\Omega L}{4b^2} \left( \frac{ad^2}{4} - \frac{a^3}{4} \right) \sin 3bt - 2\Omega C_1 \cos bt - \frac{\Omega L}{b} \left( \frac{3a^3}{4} + ac^2 + \frac{ad^2}{4} \right) \cos bt \quad (13)$$

which gives on integration.

$$y = D + \frac{(V_c + 2\Omega x_0)(2\pi G\rho - \Omega^2)}{b^2} t + \frac{2\Omega C_2}{b} \sin bt \\ + \frac{\Omega Lacd}{3b^3} \cos 2bt + \frac{\Omega L}{12b^3} \left( \frac{ad^2}{4} - \frac{a^3}{4} \right) \cos 3bt \\ - \frac{2\Omega C_1}{b} \sin bt - \frac{\Omega L}{4} \left( \frac{3a^3}{4} + ac^2 + \frac{ad^2}{4} \right) \left( \frac{t \sin bt}{b} + \frac{\cos bt}{b^2} \right) \quad (14)$$

To determine the arbitrary constant  $D$ , we apply the initial conditions,  $t=0$ ,  $y = R_1$  in equation (14), on simplification, we get

$$D = R_1 + \frac{\Omega La}{3b^3} (4a^2 + 6c^2 + 2d^2 + 3cd) \quad (15)$$

The orbit of the gas element will be given by the equations (11) and (14). The help of relations (7), (12) and (15) will also be required.

Perhaps, it will not be out of place to mention if we put  $L=0$  then the equations reduces to that for unrestricted motion.

**Case II :**

We will now consider motion taking the resistance to vary as the fifth power of the velocity. The equations of motion are

$$\left. \begin{aligned} \frac{d^2x}{dt^2} - 2\Omega \frac{dy}{dt} - \Omega^2 x &= X_c - Lv^5 \cos \psi \\ \frac{d^2y}{dt^2} + 2\Omega \frac{dx}{dt} - \Omega^2 y &= Y_c - Lv^5 \sin \psi \end{aligned} \right\} \quad (16)$$

Substituting  $V_c - Lv^5 \sin \psi + \Omega^2 y = 0$ , in above equation, we have

$$\frac{d^2 y}{dt^2} = -2 \Omega \frac{dx}{dt}$$

which yields on integration, and applying initial conditions

$$\frac{dy}{dt} = V_c - 2 \Omega (x - x_0) \quad (17)$$

Using equations (16), and (17), and on simplification

$$\frac{d^2 x}{dt^2} + (2 \pi G \rho + 3 \Omega^2) x = (2 \Omega V_c + 4 \Omega^2 x_0) - L \left( \frac{ds}{dt} \right)^4 \cdot \frac{dx}{dt} \quad (18)$$

We have

$$\frac{dx}{dt} \cdot \left( \frac{ds}{dt} \right)^4 = a \sin bt \{a^2 \sin^2 bt + (c + d \cos bt)^2\}^2 \quad (19)$$

where as before

$$\left. \begin{aligned} a &= \frac{(\Omega^2 x_0 + 2 \Omega V_c - 2 \pi G \rho x_0)}{(2 \pi G \rho + 3 \Omega^2)^{1/2}} \\ b &= (2 \pi G \rho + 3 \Omega^2)^{1/2} \\ c &= \frac{(V_c + 2 \Omega x_0) (2 \pi G \rho - 3 \Omega^2)}{2 \pi G \rho + 3 \Omega^2} \\ d &= \frac{2 \Omega (\Omega^2 x_0 + 2 \Omega V_c - 2 \pi G \rho x_0)}{2 \pi G \rho + 3 \Omega^2} \end{aligned} \right\} \quad (20)$$

and

On simplification

$$\begin{aligned} \frac{dx}{dt} \cdot \left( \frac{ds}{dt} \right)^4 &= \frac{a}{8} (5a^4 + 8c^4 + 12c^2 d^2 + d^4 + 12a^2 c^2 + 2a^2 d^2) \sin bt + \\ &+ acd (a^2 + 2c^2 + d^2) \sin 2bt \\ &+ \frac{a}{16} (-5a^4 + 24c^2 d^2 + 3d^4 - 8a^2 c^2 + 2a^2 d^2) \sin 3bt \\ &+ \frac{acd}{2} (d^3 - a^3) \sin 4bt + \frac{a}{16} (a^2 - d^2)^2 \sin 5bt \end{aligned} \quad (21)$$

From equations (18), (20) and (21)

$$\begin{aligned} \frac{d^2 x}{dt^2} + b^2 x &= (2 \Omega V_c + 4 \Omega^2 x_0) - \frac{La}{8} (5a^4 + 8c^4 + 12c^2 d^2 + d^4 + 12a^2 c^2 + 2a^2 d^2) \sin bt \\ &- Lacd (a^2 + 2c^2 + d^2) \sin 2bt - \frac{La}{16} (-5a^4 + 24c^2 d^2 + 3d^4 - 8a^2 c^2 + 2a^2 d^2) \sin 3bt \\ &- \frac{Lacd}{2} (d^3 - a^3) \sin 4bt - \frac{La}{16} (a^2 - d^2)^2 \sin 5bt \end{aligned} \quad (22)$$

the solution of which is

$$\begin{aligned} x &= C_1 \cos bt + C_2 \sin bt + \frac{2 \Omega V_c + 4 \Omega^2 x_0}{b^2} \\ &+ \frac{La}{16b} (5a^4 + 8c^4 + 12c^2 d^2 + d^4 + 2a^2 d^2 + 12a^2 c^2) t \cos bt \end{aligned}$$

$$+ \frac{Lacd}{3b^2} (a^2 + 2c^2 + d^2) \sin 2bt + \frac{La}{128b^2} (-5a^4 + 24c^2d^2 + 3d^4 - 8a^2c^2 + 2a^2d^2) \sin 3bt + \frac{Lacd}{30b^2} (d^2 - a^2) \sin 4bt + \frac{La}{384b^2} (a^2 - d^2)^2 \sin 5bt \quad (23)$$

where  $C_1$  and  $C_2$  can be determined from the initial conditions

$$\left. \begin{aligned} C_1 &= -\frac{a}{b^2} \\ C_2 &= -\frac{La}{240b^2} \{50a^4 + 120c^4 - 315c^2d^2 + 35d^4 - 35a^2d^2 \\ &\quad + 135a^2c^2 + 128a^2cd + 320c^3d + 192cd^3\} \end{aligned} \right] \quad (24)$$

From equations (17) and (23), on simplification, we have

$$\begin{aligned} &\frac{dy}{dt} \frac{(V_c + 2\Omega x_0) 2\pi G\rho - \Omega^2}{b^2} - 2\Omega C_2 \sin bt \\ &- \frac{2\Omega Lacd}{3b^2} (a^2 + 2c^2 + d^2) \sin 2bt - \frac{\Omega La}{64b^2} (-5a^4 + 24c^2d^2 + 3d^4 - 8a^2c^2 + 2a^2d^2) \sin 3bt \\ &- \frac{\Omega Lacd}{15b^2} (d^2 - a^2) \sin 4bt - \frac{\Omega La}{192b^2} (a^2 - d^2)^2 \sin 5bt \\ &- 2\Omega C_1 \cos bt - \frac{\Omega La}{8b} (5a^4 + 8c^4 + 12c^2a^2 + d^4 + 2a^2d^2 + 12a^2c^2) t \cos bt \end{aligned} \quad (25)$$

which on integration gives

$$\begin{aligned} y &= D + \frac{(V_c + 2\Omega x_0) (2\pi G\rho - \Omega^2)}{b^2} t + \frac{2\Omega C_2}{b} \cos bt \\ &+ \frac{\Omega Lacd}{3b^3} (a^2 + 2c^2 + d^2) \cos 2bt - \frac{\Omega La}{192b^3} (-5a^4 + 24c^2d^2 + 3d^4 - 8a^2c^2 \\ &+ 2a^2d^2) \cos 3bt + \frac{\Omega Lacd}{60b^3} (d^2 - a^2) \cos 4bt + \frac{\Omega La (a^2 - d^2)^2}{960b^3} \cos 5bt \\ &- \frac{2\Omega C_1}{b} \sin bt - \frac{\Omega La}{8b} (5a^4 + 8c^4 + 12c^2a^2 + d^4 + 2a^2d^2 + 12a^2c^2) \\ &\quad \left( \frac{t \sin bt}{b} + \frac{\cos bt}{b^2} \right) \end{aligned} \quad (26)$$

where  $D$  will be given, using initial conditions, in above equation, on simplification, by

$$D = R_1 + \frac{\Omega La}{60b^3} (64a^4 + 120c^4 + 240c^2d^2 + 24d^4 - 3a^2d^2 + 160a^2c^2 + 45a^2cd + 120c^3d + 75cd^3) \quad (27)$$

The orbit of the gas element will be given by equations (23) and (26) and the relations (20), (24) and (27).

It may be seen, if  $L = 0$ , i.e., if effect due to resistance be neglected, we get the equations when resistance is not considered.

It will be noted the above investigation is valid if we assume the resistance to vary as the  $n^{th}$  power of the velocity, where  $n$  is an odd integer. We have discussed the case for  $n = 3$  and 5. Higher values of  $n$  will only involve laborious steps. We are at present reluctant to digress into detailed mathematical analysis unless substantiated by physical evidences, as well.

*Case III :*

We will now determine motion taking a different type of force law. We will assume the resistance to vary as  $v^p \cos^{p-1} \psi$ , where  $p$  is a positive integer and where  $\psi$  is the angle which the tangent makes with the  $x$ -axis. The equations of motion are thus

$$\left. \begin{aligned} \frac{d^2x}{dt^2} - 2\Omega \frac{dy}{dt} - \Omega^2 x &= X_c - Lv^p \cos^{p-1} \psi \cdot \cos \psi \\ \frac{d^2y}{dt^2} + 2\Omega \frac{dx}{dt} - \Omega^2 y &= Y_c - Lv^p \cos^{p-1} \psi \cdot \sin \psi \end{aligned} \right\} \quad (28)$$

Substituting  $Y_c - Lv^p \cos^{p-1} \psi \cdot \sin \psi + \Omega^2 y = 0$  in above equation

$$\frac{d^2y}{dt^2} = -2\Omega \frac{dx}{dt}$$

which gives on integration, and applying initial condition

$$\frac{dy}{dt} = V_c - 2\Omega (x - x_0) \quad (29)$$

Using equations (28) and (29), and simplifying

$$\frac{d^2x}{dt^2} + (2\pi G\rho + 3\Omega^2) x = (2\Omega V_c + 4\Omega^2 x_0) - L \left( \frac{dx}{dt} \right)^p \quad (30)$$

We have

$$\frac{dx}{dt} = a \sin bt \quad (31)$$

where, as before

$$\left. \begin{aligned} a &= \frac{(\Omega^2 x_0 + 2\Omega V_c - 2\pi G\rho x_0)}{(2\pi G\rho + 3\Omega^2)^{1/2}} \\ b &= (2\pi G\rho + 3\Omega^2)^{1/2} \end{aligned} \right\} \quad (32)$$

From equations (30), (31) and (32), we have

$$\frac{d^2x}{dt^2} + b^2 x = (2\Omega V_c + 4\Omega^2 x_0) - La^p \sin^p bt \quad (33)$$

(i) when  $p$  is an even positive integer :

In case  $p$  has an even positive values, equation (33) can be written as

$$\left. \begin{aligned} \frac{d^2x}{dt^2} + b^2 x &= (2\Omega V_c + 4\Omega^2 x_0) - \frac{La^p}{2^{p-1} (-1)^{p/2}} \left\{ \cos p bt \right. \\ &\quad \left. - p \cos (p-2) bt + \frac{p(p-1)}{1-2} \cos (p-4) bt - \dots + \frac{1}{2} (-1)^{p/2} p C_{p/2} \right\} \end{aligned} \right\} \quad (34)$$

the solution of which is

$$x = C_1 \cos bt + C_2 \sin bt + \frac{2\Omega V_c + 4\Omega^2 x_0}{b^2} - \frac{La^p}{2^{p-1} (-1)^{p/2}}$$

$$\left[ \begin{aligned} & \frac{1}{b^2\{1-p^2\}} \cdot \cos pbt - \frac{p}{b^2\{1-(p-2)^2\}} \cdot \cos (p-2)bt \\ & + \frac{p(p-1)}{1 \cdot 2} \cdot \frac{1}{b^2\{1-(p-4)^2\}} \cdot \cos (p-4)bt \\ & - \frac{p(p-1)(p-2)}{1 \cdot 2 \cdot 3} \cdot \frac{1}{b^2\{1-(p-6)^2\}} \cdot \cos (p-6)bt + \dots \\ & + \frac{1}{2}(-1)^{p/2} \frac{{}^p C_{p/2}}{b^2} \end{aligned} \right] \quad (35)$$

where the arbitrary constants  $C_1$  and  $C_2$  can be evaluated, from the initial conditions, as

$$\begin{aligned} C_1 = & -\frac{a}{b^2} + \frac{La^p}{b^2 \cdot 2^{p-1}(-1)^{p/2}} \left[ \frac{1}{\{1-p^2\}} - \frac{p}{\{1-(p-2)^2\}} + \frac{p(p-1)}{1 \cdot 2} \cdot \frac{1}{\{1-(p-4)^2\}} \right. \\ & \left. - \frac{p(p-1)(p-2)}{1 \cdot 2 \cdot 3} \cdot \frac{1}{\{1-(p-6)^2\}} + \dots + \frac{1}{2}(-1)^{p/2} {}^p C_{p/2} \right] \\ C_2 = & 0 \end{aligned} \quad (36)$$

From equation (29) and (35), on simplification, we find

$$\begin{aligned} \frac{dy}{dt} \frac{(V_c + 2\Omega x_0)(2\pi G\rho - \Omega^2)}{b^2} - 2\Omega C_1 \cos bt - 2\Omega C_2 \sin bt \\ + \frac{2\Omega La^p}{2^{p-1}(-1)^{p/2}} \left[ \frac{1}{b^2\{1-p^2\}} \cdot \cos pbt - \frac{p}{b^2\{1-(p-2)^2\}} \cdot \cos (p-2)bt \right. \\ + \frac{p(p-1)}{1 \cdot 2} \cdot \frac{1}{b^2\{1-(p-4)^2\}} \cdot \cos (p-4)bt - \frac{p(p-1)(p-2)}{1 \cdot 2 \cdot 3} \cdot \frac{1}{b^2\{1-(p-6)^2\}} \cdot \\ \left. \cos (p-6)bt + \dots + \frac{1}{2}(-1)^{p/2} \frac{{}^p C_{p/2}}{b^2} \right] \end{aligned} \quad (37)$$

which gives on integration

$$\begin{aligned} y = & D + \frac{(V_c + 2\Omega x_0)(2\pi G\rho - \Omega^2)}{b^2} t - \frac{2\Omega C_1}{b} \sin bt \\ & + \frac{2\Omega C_2}{b} \cos bt + \frac{2\Omega La^p}{2^{p-1}(-1)^{p/2}} \left[ \frac{1}{b^2 \cdot \{1-p^2\} \cdot pb} \cdot \sin pbt \right. \\ & - \frac{p}{b^2 \cdot \{1-(p-2)^2\} \cdot (p-2)b} \cdot \sin (p-2)bt + \frac{p(p-1)}{1 \cdot 2} \cdot \frac{1}{b^2 \cdot \{1-(p-4)^2\} \cdot (p-4)b} \cdot \\ & \sin (p-4)bt - \frac{p(p-1)(p-2)}{1 \cdot 2 \cdot 3} \cdot \frac{1}{b^2 \cdot \{1-(p-6)^2\} \cdot (p-6)b} \cdot \sin (p-6)bt + \dots \\ & \left. + \frac{1}{2}(-1)^{p/2} \frac{{}^p C_{p/2}}{b^2} t \right] \end{aligned} \quad (38)$$

Applying the initial conditions,  $t = 0, y = R_1$ , in equation (38), we find

$$D = R_1 \quad (39)$$

The orbit of the gas element will be given by equations (35) and (38), and the relations (32), (36) and (39).

If  $L = 0$  we get the equations when effect due to resistance is neglected.

(ii) when  $p$  is an odd positive integer :

When  $p$  is positive, and an odd number, equation (33) can be written as

$$\frac{d^2x}{dt^2} + b^2x = (2\Omega V_c + 4\Omega^2x_0) - \frac{La^p}{2^{p-1}(-1)^{(p-1)/2}} \left[ \sin pbt - p \sin (p-2)bt + \frac{p(p-1)}{1 \cdot 2} \sin (p-4)bt + \dots + (-1)^{(p-1)/2} p C_{(p-1)/2} \sin bt \right] \quad (40)$$

The solution of which is

$$x = C_1 \cos bt + C_2 \sin bt + \frac{2\Omega V_c + \Omega^2x_0}{b^2} - \frac{La^p}{2^{p-1}(-1)^{(p-1)/2}} \left[ \frac{1}{b^2 \{1-p^2\}} \sin pbt - \frac{p}{b^2 \{1-(p-2)^2\}} \sin (p-2)bt + \frac{p(p-1)}{1 \cdot 2} \cdot \frac{1}{b^2 \{1-(p-4)^2\}} \sin (p-4)bt - \frac{p(p-1)(p-2)}{1 \cdot 2 \cdot 3} \cdot \frac{1}{b^2 \{1-(p-6)^2\}} \sin (p-6)bt + \dots + (-1)^{(p-1)/2} p C_{(p-1)/2} \cdot \frac{\cos bt}{(-2b)} \right] \quad (41)$$

where  $C_1$  and  $C_2$ , in above equation, are arbitrary constants, the value of which is determined, from the initial conditions, as follows :

$$C_1 = -\frac{a}{b^2} \left\{ \begin{aligned} C_2 = & \frac{La^p}{b^2 2^{p-1}(-1)^{(p-1)/2}} \left[ \frac{p}{\{1-p^2\}} \cdot \frac{p}{\{1-(p-2)^2\}} \cdot (p-2) \right. \\ & + \frac{p(p-1)}{1 \cdot 2} \cdot \frac{(p-4)}{\{1-(p-4)^2\}} - \frac{p(p-1)(p-2)}{1 \cdot 2 \cdot 3} \cdot \frac{(p-6)}{\{1-(p-6)^2\}} \\ & \left. + \dots + (-1)^{(p-1)/2} p C_{(p-1)/2} \right] \end{aligned} \right\} \quad (42)$$

From equations (29) and (41), we find on simplification

$$\frac{dy}{dt} = \frac{(V_c + 2\Omega x_0)(2\pi G\rho - \Omega^2)}{b^2} - 2\Omega C_1 \cos bt - 2\Omega C_2 \sin bt + \frac{2\Omega La^p}{2^{p-1}(-1)^{(p-1)/2}} \left[ \frac{1}{\{1-p^2\} \cdot b^2} \sin pbt - \frac{p}{\{1-(p-2)^2\} \cdot b^2} \sin (p-2)bt + \frac{p(p-1)}{1 \cdot 2} \cdot \frac{1}{\{1-(p-4)^2\} \cdot b^2} \sin (p-4)bt - \frac{p(p-1)(p-2)}{1 \cdot 2 \cdot 3} \cdot \frac{1}{\{1-(p-6)^2\} \cdot b^2} \sin (p-6)bt + \dots + (-1)^{(p-1)/2} p C_{(p-1)/2} \frac{\cos bt}{(-2b)} \right] \quad (43)$$

which gives on integration.

$$\begin{aligned}
 y = & D + \frac{(V_c + 2\Omega x_0)(2\pi G\rho - \Omega^2)}{b^2} t - \frac{2\Omega C_1}{b} \sin bt + \frac{2\Omega C_2}{b} \cos bt \\
 & - \frac{2\Omega La^p}{b^2 \cdot 2^{p-1} (-1)^{(p-1)/2}} \left[ \frac{1}{\{1-p^2\} \cdot pb} \cdot \cos bt - \frac{p}{\{1-(p-2)^2\} \cdot (p-2)b} \cdot \right. \\
 & \cos (p-2) bt + \frac{p(p-1)}{1 \cdot 2} \cdot \frac{1}{\{1-(p-4)^2\} \cdot (p-4)b} \cdot \cos (p-4) bt - \frac{p(p-1)(p-2)}{1 \cdot 2 \cdot 3} \\
 & \cdot \frac{1}{\{1-(p-6)^2\} \cdot (p-6)b} \cdot \cos (p-6) bt + \dots + (-1)^{(p-1)/2} p C_{(p-1)/2} \\
 & \left. \left( \frac{bt \sin bt + \cos bt}{-2b} \right) \right] \quad (44)
 \end{aligned}$$

The value of  $D$ , after some simplification, is

$$\begin{aligned}
 D = & R_1 + \frac{2\Omega La^p}{b^3 \cdot 2^{p-1} (-1)^{(p-1)/2}} \left[ \frac{1}{p} - \frac{p}{p-2} \right. \\
 & \left. + \frac{p(p-1)}{1 \cdot 2 \cdot (p-4)} - \frac{p(p-1)(p-2)}{1 \cdot 2 \cdot 3 \cdot (p-6)} + \dots \right] \quad (45)
 \end{aligned}$$

The orbit of the gas element will be given by equations (41) and (44) and the relations (32), (42) and (45).

As before, if  $L=0$ , we get the equations when resistance is not considered.

Perhaps, it will not be out of place to mention, if instead we consider, as uniformly rotating, homogeneous elliptic cylinder model for the barred galaxy, then the investigations will also be valid for all the resistance laws, and the equations will be tractable. Only the value of the constants will be suitably modified.

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## The influence of Phosphates on the slow oxidation of Cellulosic organic matter and Coal

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### Abstract

Influence of different energy materials (1.0% carbon) *viz.*, Wheat straw, Banz leaves and Coal in combination with phosphates (0.5%  $P_2O_5$ ) like Trichinopoly and Bihar rock phosphates and German and Tata Basic slags were studied with Srinagar (Garhwal) soil of U. P.

Energy materials when added to soil are slowly oxidised and considerable increase in total nitrogen content of the system takes place. The efficiency of nitrogen fixation is always nearly double in light than in the dark, suggesting that light energy absorbed by the system is also utilized in accomplishing the endothermal reaction of ammonia formation. There is an appreciable increase in the availability of phosphate when energy materials are allowed to undergo slow oxidation on the soil surface. The increase in available phosphate is enhanced considerably when phosphates are added. It is found that the amount of carbon oxidised as well as an increase in nitrogen fixed with energy materials are in the following order .

Wheat straw > Banz leaves > Coal.

It is further observed that the oxidation of carbon and fixation of nitrogen is greater with basic slags than with the rock phosphates.

They are in the following order :

G. B. S. > T. B. S. > T. R. P. > B. R. P.

### Introduction

Phosphorus, besides being closely associated with nitrogen transformations, is itself a vital constituent of soil fertility. Collings<sup>1</sup> has reported a better growth of Wheat crop when its residues were mixed with rock phosphates and ploughed, in comparison with when only crop was ploughed in. Ralph and Weibling<sup>2</sup> found that by phosphating Bernard clay loam with superphosphate, basic slags and rock phosphate increased yields more than double in comparison to unphosphated plot were obtained. Many workers<sup>3-5</sup> found that organic matter increased the availability of soil phosphate and of rock phosphate which was added to the soil as fertilizer. Dhar and Mukherji<sup>6</sup> have shown that when cane sugar is added to soils and exposed to light, there is an appreciable increase in the nitrogen content. Bjalfve<sup>7</sup> in Sweden has also found that straw, when mixed with soil or sand, fixes atmospheric nitrogen and also sun light markedly increases the nitrogen fixation by the slow oxidation of straw in sand or soil.



## Experimental

Experiments were carried out with soil collected from a cultivated farm of Srinagar, Garhwal, U. P. 200 gms. of soil, after passing through a 70 mesh sieve, were taken in clean enamelled dishes. To this, required amount of phosphates—Tata and German basic slags and Trichinopoly and Bihar rock phosphates each equivalent to 0.5% phosphoric acid were added. Carbonaceous materials—Wheat straw, Banz leaves and Bituminous coal each equivalent to 1.0% carbon were added and the contents were thoroughly mixed. Two similar sets were arranged side by side, one of which was exposed to light under a 500 Watt bulb and other was covered with a thick black cloth to cut off light. In all the dishes, the moisture content was maintained at 30% level throughout the experiment. The contents of the dishes were stirred with a glass rod from time to time to facilitate the oxidation of energy materials. The light exposure was limited to an average of 8 hours a day.

### *Analysis of Soil and Energy materials used*

Estimation	Soil%	Wheat straw%	Banz leaves%	Coal%
Loss on ignition	8.5728	88.6072	90.2432	87.2432
Ash		10.2500	7.5898	12.2026
HCl insoluble	79.1052	4.8824	3.2562	6.4030
Fe <sub>2</sub> O <sub>3</sub>	3.0243	0.8255	1.3421	0.8580
CaO	0.6034	0.7850	0.7210	1.2020
MgO	0.3607	0.4033	0.1866	1.2018
Total P <sub>2</sub> O <sub>5</sub>	0.0842	0.6248	0.6892	0.3110
Available P <sub>2</sub> O <sub>5</sub>	0.02156			
K <sub>2</sub> O	1.2340	0.8202	1.4124	0.8881
Total Carbon	0.4406	38.4856	38.0240	62.0020
Total Nitrogen	0.04016	0.63116	2.0922	1.5820
NH <sub>3</sub> -N	0.0025			
NO <sub>3</sub> -N	0.0028			
C/N ratio	10.97	60.9	18.2	39.2

### *Analysis of Phosphates*

Estimation	Basic slags		Rock phosphates	
	German basic slag%	Tata basic slag%	Trichinopoly rock phosphate%	Bihar rock phosphate%
SiO <sub>2</sub>	11.5820	16.3420	14.2140	15.5210
Fe <sub>2</sub> O <sub>3</sub>	13.9846	12.9298	4.4380	3.9400
CaO	42.7850	37.9684	20.8000	13.7980
MgO	5.2346	4.9736	1.0278	0.4780
MnO	5.9736	2.9076		
K <sub>2</sub> O	Traces	0.6481	0.7040	0.6012
Total P <sub>2</sub> O <sub>5</sub>	17.4600	7.1402	29.6000	30.0000
Available P <sub>2</sub> O <sub>5</sub>	9.9400	4.0436	1.4500	1.3021

*The following indicates the balance of Nitrogen and Phosphoric acid*

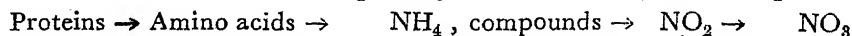
Treatments	Total carbon unoxidised (gms)	Total carbon oxidised (gms)	Initial nitrogen (gms)	Total nitrogen (gms)	Increase in nitrogen (gms)	Efficiency	Initial Avail. $P_2O_5$ (gms.)	Total $P_2O_5$ (gms)
<i>Wheat straw group</i>								
200 gms. Soil								
Light	1.6212	1.2600	0.1132	0.1515	0.0383	30.4	0.0431	0.0508
Dark	1.9602	0.9210	0.1132	0.1285	0.0153	16.4	0.0431	0.0482
200 gms. Soil + 0.5% $P_2O_5$ as T. R. P.	1.5312	1.3500	0.1132	0.1712	0.0580	42.9	0.0927	0.1265
Light	1.9042	0.9770	0.1132	0.1352	0.0220	22.4	0.0927	0.1130
200 gms. Soil + 0.5% $P_2O_5$ as B. R. P.	1.5655	1.3157	0.1132	0.1632	0.0500	28.0	0.0564	0.1098
Light	1.9232	0.9580	0.1132	0.1320	0.0188	19.6	0.0864	0.1001
200 gms. Soil + 0.5% $P_2O_5$ as T. B. S.	1.4835	1.3977	0.1132	0.1740	0.0608	43.5	0.0694	0.0629
Light	1.8800	1.0012	0.1132	0.1366	0.0234	23.3	0.0694	0.0649
200 gms. Soil + 0.5% $P_2O_5$ as G. B. S.	1.4701	1.4111	0.1132	0.1767	0.0635	45.0	0.06124	0.0685
Light	1.8639	1.0173	0.1132	0.1376	0.0244	23.9	0.06124	0.06501
Dark								
<i>Banz leaves group</i>								
200 gms. Soil								
Light	1.6512	1.2300	0.1902	0.2265	0.0363	29.5	0.0431	0.0504
Dark	1.9652	0.9160	0.1902	0.2052	0.0150	16.3	0.0431	0.0479
200 gms. Soil + 0.5% $P_2O_5$ as T. R. P.	1.5662	1.3150	0.1902	0.2452	0.0350	41.5	0.0927	0.1258
Light	1.9142	0.9670	0.1902	0.2115	0.0213	22.0	0.0927	0.1125
200 gms. Soil + 0.5% $P_2O_5$ as B. R. P.	1.5862	1.2950	0.1902	0.2376	0.0474	36.6	0.0864	0.1091
Light	1.9352	0.9460	0.1902	0.2086	0.0183	19.3	0.0864	0.0995
200 gms. Soil + 0.5% $P_2O_5$ as T. B. S.	1.5400	1.3412	0.1902	0.2468	0.0566	42.2	0.0694	0.0622
Light	1.8892	0.9920	0.1902	0.2130	0.0228	29.2	0.0694	0.0643
200 gms. Soil + 0.5% $P_2O_5$ as G. B. S.	1.5267	1.3545	0.1902	0.2498	0.0596	44.0	0.06124	0.0677
Light	1.8692	1.0120	0.1902	0.2142	0.0240	23.7	0.06124	0.06495
Dark								
<i>Coal group</i>								
200 gms. Soil								
Light	2.6134	0.2678	0.1311	0.1356	0.0045	16.8	0.0431	0.0459
Dark	2.6767	0.2045	0.1311	0.1329	0.0018	8.8	0.0431	0.0445
200 gms. Soil + 0.5% $P_2O_5$ as T. R. P.	2.5918	0.2894	0.1311	0.1377	0.0066	22.8	0.0927	0.1024
Light	2.6562	0.2250	0.1311	0.1338	0.0027	12.0	0.0927	0.0992
200 gms. Soil + 0.5% $P_2O_5$ as B. R. P.	2.6067	0.2745	0.1311	0.1367	0.0056	20.4	0.0864	0.0958
Light	2.6600	0.2212	0.1311	0.1336	0.0025	11.3	0.0864	0.0927
200 gms. Soil + 0.5% $P_2O_5$ as T. B. S.	2.5838	0.2974	0.1311	0.1380	0.0069	23.2	0.0694	0.0636
Light	2.6393	0.2419	0.1311	0.1341	0.0030	12.4	0.0694	0.06171
200 gms. Soil + 0.5% $P_2O_5$ as G. B. S.	2.5821	0.2991	0.1311	0.1384	0.0073	24.5	0.06124	0.06274
Light	2.6214	0.2598	0.1311	0.1344	0.0033	12.7	0.06.24	0.06225
Dark								

## Discussion

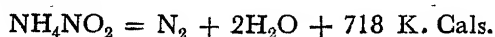
The results recorded in the foregoing tables reveal that carbonaceous materials like Wheat straw, Banz leaves and coal when added to the soil, are slowly oxidised in contact with air and there is an increase in the total nitrogen content of the system. This is possible if the energy released during the process of carbon oxidation is actually utilized in fixing atmospheric nitrogen. It is observed that when the system is illuminated, the rate of oxidation of carbonaceous compounds and the increase of nitrogen fixation are markedly enhanced, showing thereby an appreciable influence of light on the phenomenon of nitrogen fixation. Moreover, the efficiency of nitrogen fixation is always in a ratio of approximately 2:1 in light/dark, showing that the light energy absorbed by the system is also utilized in this mechanism accomplishing the endothermal reaction of ammonia formation.

The remarkable influence of light intensity and phosphate on nitrogen fixation is explained on following consideration.

Many workers<sup>8,9</sup> have conclusively proved that in soil the process of nitrogen fixation is always retarded by ammonification and nitrification. Thus, on the one hand, the nitrogen fixed on the surface of the soil undergoes various changes aided by sun light absorption to form amino acids and proteins, synthesized in the soil, undergo ammonification and nitrification which are also accelerated by light absorption and form nitrite and nitrate as outlined below :



In these series of reactions the unstable substance ammonium nitrite is formed which undergoes decomposition liberating energy, nitrogen gas and water as in the following equation :



When phosphates were added either in the form of basic slags or rock phosphates to the system, nitrogen increases in the system and in the presence of phosphates, more or less stable phosphoproteins are formed. These compounds seem to resist the process of ammonification and nitrification and hence conserve proteins and other nitrogenous compounds and thereby the nitrogen status of the soil increases.

Further, it is well known that the decomposition of ammonium nitrite an unstable intermediate product is accelerated in presence of acids. When energy materials undergo slow oxidation, some organic acids are produced in the system, there is greater nitrogen loss. But when calcium phosphates are introduced in the system, they act as buffers due to the fact that the second and third dissociation constants of phosphoric acid are small and thus the concentration of H ion in the system cannot increase much. Hence the rate of ammonification and subsequent nitrogen loss is retarded and the efficiency of nitrogen fixation is greater with phosphates as compared to that in their absence.

It has been observed that the oxidation of carbon as well as the efficiency of nitrogen fixation as also the amount of nitrogen fixed are greater with basic slags than with rock phosphate from all the energy materials. They are in the following descending order :



The basic slags are more alkaline and the alkalinity favours oxidation. Moreover, Mn, V, Fe present in traces in basic slags are helpful in oxidation.

The other important observation of great practical value is the appreciable increase in the availability of phosphate when organic materials are allowed to undergo slow oxidation on soil surface. The increase of available phosphate in the soil is enhanced considerably when phosphates are incorporated with organic matter.

Slow oxidation of organic matter produces carbonic acid, which plays a significant role in soils. It is well known that carbonic acid is much weaker than phosphoric acid. The dissociation constant of carbonic acid is smaller than that of phosphoric acid, calcium carbonate is more alkaline in its properties than calcium phosphate. Secondly, because the second and third dissociation constants of phosphoric acid are small, whenever  $H^+$  ions are available in soil solution along with phosphate ions:  $HPO_4$  and  $H_2PO_4$  ions are readily formed. Thus  $H^+$  ion concentration of the systems decreases and tricalcium phosphate, like calcium carbonate, can act as excellent buffer. Hence, phosphate also helps in decreasing the lime requirement of soils. Therefore, land fertility depends a good deal on its calcium phosphate status, because lands rich in calcium phosphate can fix atmospheric nitrogen by slow oxidation of added organic matter. The organic matter which is naturally produced undergoes decomposition leading to humus formation and can maintain buffering action of soil and check the leaching of lime. This is in agreement with the important observations of Arrhenius<sup>10</sup>, showing that soil near human habitations, past and present in different parts of Sweden and Java are rich in phosphate and are fertile. It is desirable to increase the calcium phosphate reserves of soils by adding cheap phosphating materials, like basic (Thomas) slags and rock phosphates.

It is interesting to note that the total nitrogen in the soils has been found in these experiments to be much greater in light than in the dark. These observations support the theory of photofixation as sponsored by Dhar and co-workers in India.

From these investigations, we have concluded that the nitrogen status of our soils, consequently the humus capital can be improved by ploughing in organic matter and phosphates.

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## On the Growth of Solutions of Complex Differential Systems

By

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### Abstract

In this paper, we shall derive some boundedness properties of differential systems in the complex domain by utilizing the comparison principle with the help of integral inequalities in the real domain.

### 1. Introduction

The objective of this paper is to utilize the comparison principle to determine certain boundedness properties of complex differential systems. The comparison principle has been successfully used by many authors to discuss properties of ordinary differential systems. Our results are presented in a unified way which includes as a special case the recent paper of K. M. Das[2] in which he considers a second order complex differential equation and compares the growth of the moduli of the solutions of that equation in terms of real valued continuous functions.

### 2. Preliminaries

We shall consider a complex differential system of the form

$$\frac{dy_i}{dz} = F_i(y_1, y_2, \dots, y_n, z); y_i(0) = a_i, i = 1, 2, \dots, n \quad (2.1)$$

where the  $F_i$ 's are complex valued functions of the complex variables  $y_i$ 's and  $z$ , which are regular analytic in  $z$  for  $|z| < R$ . We shall assume that there exist functions  $w_i(u_i, u_2, \dots, u_n, x)$  which are continuous, non-decreasing in  $u_i$  and bounded in the region  $0 \leq x < \gamma$  for  $\gamma \leq R$  and  $u_i > 0$ . Let  $m_i(x)$  represent the maximum solution of the differential system

$$u_i' = w_i(u_1, u_2, \dots, u_n, x), u_i(0) = a_i. \quad (2.2)$$

It has been shown by Kamke, [3], that under the above assumptions the maximum and minimum solutions of the differential system exist. During the development of the objectives of this paper we shall utilize the following lemma, the proof of which can be found in [4].

*Lemma :* Let the functions  $w_i(u_1, u_2, \dots, u_n, x) \geq 0$ , be continuous, bounded and non-decreasing in  $u_i$  in the region  $0 \leq x < \infty, |u_i| < \infty$ . Also the functions  $u_i(x)$  are continuous in the region  $0 \leq x < \infty$  and satisfy the following inequality :

$$u_i(x) \leq a_i + \int_0^x w_i(s, u_1, u_2, \dots, u_n) ds, \quad u_i(0) = a_i.$$

Then,

$$u_i(x) \leq m_i(x), \quad 0 \leq x < \infty$$

where  $m_i(x)$  is the maximal solution of the differential system (2.2).

### 3. Main results

Relative to the objectives of this paper we state and prove the following theorems.

**Theorem 1:** Let

(i) the functions  $w_i(u_1, u_2, \dots, u_n, x)$  be continuous, non-decreasing in  $u_i$  and bounded in the region  $0 \leq x < \gamma$  for  $\gamma \leq R$  and  $u_i > 0$ .

(ii) the functions  $F_i$  of the complex differential system (2.1) satisfy the following condition:

$$|F_i(y_1, y_2, \dots, y_n, z)| \leq w_i(|y_1|, |y_2|, \dots, |y_n|, |z|) \quad \text{for all } y_i \text{ and } |z| < R.$$

Then,

$$|y_i(z)| \leq m_i(x)$$

whenever,  $|z| \leq m_i(0)$ ,  $i = 1, 2, \dots, n$ .

*Proof:* Let  $y_i(z)$  be the non-zero solution of the complex differential system (2.1).

Let  $z = xe^{i\theta}$ ,  $\bar{y}_i(x, \theta) = y_i(xe^{i\theta})$ , and  $u_i = |\bar{y}_i|$ . Then

$$\frac{\partial \bar{y}_i}{\partial x}(x, \theta) = \frac{dy_i}{dz}(xe^{i\theta}) \cdot e^{i\theta}$$

and thus

$$\begin{aligned} \left| \frac{\partial \bar{y}_i}{\partial x}(x, \theta) \right| &= \left| \frac{dy_i}{dz}(xe^{i\theta}) \right| \\ &= |F_i(y_1, y_2, \dots, y_n, z)| \\ &\leq w_i(|y_1|, |y_2|, \dots, |y_n|, x). \end{aligned}$$

However, we know, [1], that

$$\left| \frac{\partial u_i}{\partial x} \right| \leq \left| \frac{\partial \bar{y}_i}{\partial x} \right|.$$

Hence,

$$\left| \frac{\partial u_i}{\partial x} \right| \leq w_i(u_1, u_2, \dots, u_n, x).$$

Integrating the above inequality from 0 to  $x$ , we obtain

$$u_i(x) - u_i(0) \leq \int_0^x w_i(u_1, u_2, \dots, u_n, s) ds,$$

that is,

$$u_i(x) \leq u_i(0) + \int_0^x w_i(u_1, u_2, \dots, u_n, s) ds$$

$$\leq a_i + \int_0^x w_i(u_1, u_2, \dots, u_n, s) ds.$$

Applying the lemma which we have introduced we obtain

$$u_i(x) \leq m_i(x), i = 1, 2, \dots, n,$$

that is,

$$|y_i(z)| \leq m_i(x), i = 1, 2, \dots, n.$$

This completes the proof of the theorem. In the following theorems we assume that the solutions of the complex differential system (2.1) exist for  $|z| < \infty$ .

**Theorem 2:** Suppose that the assumptions of Theorem 1 hold. Let the maximum solution  $m_i(x)$  of (2.2) satisfy the following condition:

$$\lim_{x \rightarrow \infty} m_i(x) < \infty, i = 1, 2, \dots, n.$$

Then, any solution of (2.1) for which  $|y_i(0)| = a_i$  is regular for  $|z| < R$ .

**Theorem 3:** Suppose that the assumptions of Theorem 1 hold. Assume further that the maximum solution  $m_i(x) = 0$  ( $L_i(x)$ ) as  $x \rightarrow \infty$ , where  $L_i(x)$  are continuous and positive functions in  $0 \leq x < \infty$ . Then every solution  $y_i(z)$  of the complex differential system (2.1) satisfies the following relation

$$|y_i(x)| = 0 (L_i(x)), \text{ as } x \rightarrow \infty \text{ and } |z| = x.$$

The proof of the above theorems are direct consequences of Theorem 1 and are omitted.

4. We now proceed to show that the results obtained in [2] are special cases of the above theorems.

Consider the differential equation of the form

$$\frac{d^2 y_i}{dz^2} = F(y_1, z) \quad (4.1)$$

where  $F(y_1, z)$  is the entire function of  $y_1$  which is regular analytic in  $z$  for  $|z| < R$ . Assume that

$$|F(y_1, z)| \leq w(|y_1|, |z|)$$

for all  $y_1, |z| < R$  where the majorant  $w(s, t)$  is continuous in  $t$  for  $0 \leq t < R$  and is non-decreasing function of  $s$  for  $0 \leq s < \infty$ .

Let  $u_1(x)$  be a continuous function in  $(0, \gamma)$ ,  $\gamma \leq R$  satisfying

$$u_1'' = w(u_1(x), x). \quad (4.2)$$

Setting

$$\left. \begin{aligned} \frac{dy_1}{dz} &= y_2 \\ \frac{dy_2}{dz} &= F(y_1, z) \end{aligned} \right\} \quad (4.1)'$$

and

$$\left. \begin{aligned} u_1' &= u_2 \\ u_2' &= w(u_1, x) \end{aligned} \right\} \quad (4.2)'$$

it follows from Theorem 1 that

$$|y_1(z)| \leq u_1(x)$$

and

$$|y_2(z)| \leq u_2(x)$$

where  $(u_1, u_2)$  is a solution of the system (4.2)'. In particular, if

$$|F(y_1, z)| \leq P(z) y_1^n, n > 1$$

where  $P(z)$  satisfies the following inequality

$$|P(z)| \leq \alpha(1-|z|)^{\varepsilon-2}, \alpha, \varepsilon > 0$$

then the solutions of (4.1) are bounded in the unit disc together with its derivatives. The system (4.2)' can be reduced to the form

$$u_1' = u_2$$

$$u_2' = \frac{\alpha u_1^n}{(1-x)^{2-\varepsilon}}$$

the solutions of which are given by the equations

$$u_1 = \beta(1-x)^{-\mu}$$

$$u_2 = \mu \beta(1-x)^{-(\mu+1)}, \mu, \beta > 0$$

if  $\mu$  and  $\beta$  satisfy the relations

$$\mu(\mu+1) = \alpha \beta^{n-1}$$

and

$$\mu = n\mu - \varepsilon.$$

It is clear to see that with our approach we obtain additional information about the growth of the derivatives of the solution. Furthermore,  $u = (u_1, u_2)$  being the solution of the differential system, we obtain sharper bounds than those obtained in [2]. Finally, it is clear that from our results we can consider differential equations of the form

$$\frac{d^ny}{dz^n} = F\left(y, \frac{dy}{dz}, \frac{d^2y}{dz^2}, \dots, \frac{d^{n-1}y}{dz^{n-1}}, z\right).$$

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## Hypergeometric Functions of Three Variables

By

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### Abstract

This paper is devoted to the study of two new Hypergeometric Functions of three variables  $G_C$ ,  $G_D$  which I came across in the course of a further study of two Functions  $G_A$ ,  $G_B$  defined by R. C. Pandey.

### 1. Introduction

Lauricella [5], in the year 1893, generalised Appell's four functions of  $n$  variables. For  $n=3$ , he conjectured the existence of ten hypergeometric functions of three variables in addition to  $F_A$ ,  $F_B$ ,  $F_C$  and  $F_D$  defined by him. These ten functions  $F_E$ ,  $F_F$ ,  $F_G$ ,  $F_H$ ,  $F_I$ ,  $F_J$ ,  $F_K$ ,  $F_L$ ,  $F_M$ ,  $F_N$  were defined and their properties studied, by Saran[7].

Recently, while studying Saran's functions, Pandey[6] defined two more functions of three variables namely  $G_A$ ,  $G_B$ . In the course of a further study of these two functions, I came across two more new hypergeometric functions of three variables defined as

$$(1.1) \quad G_C\left(\frac{1}{\alpha}, \alpha_1, \alpha, \beta, \beta_1, \beta; \frac{1}{\gamma}, \gamma, \gamma; x, y, z\right) \\ = \sum \frac{(\alpha, p-m)(\alpha_1, n)(\beta, m+p)(\beta_1, n)}{(1, m)(1, n)(1, p)(\gamma, n+p-m)} x^m y^n z^p,$$

and

$$(1.2) \quad G_D\left(\frac{1}{\alpha}, \alpha_1, \alpha, \beta_1, \beta_2, \beta_3; \frac{1}{\gamma}, \gamma, \gamma; x, y, z\right) \\ = \sum \frac{(\alpha, p-m)(\alpha_1, n)(\beta_1, m)(\beta_2, n)(\beta_3, p)}{(1, m)(1, n)(1, p)(\gamma, n+p-m)} x^m y^n z^p.$$

where the series extend for positive integral values of  $m, n, p$  from zero to infinity.

The series are uniformly convergent if  $|x| < 1$ ,  $|y| < 1$ ,  $|z| < 1$ .

The functions reduce to  $F_3$  for  $x = 0$  and for  $y = 0$  to  $G_1$ , and  $G_2$  respectively.

## 2. Integral representations

From the definition of  $G_C$ ,

$$(2.1) \quad G_C\left(\frac{1}{\alpha}, \alpha_1, \alpha, \beta, \beta_1, \beta; \frac{1}{\gamma}, \gamma, \gamma; x, y, z\right) \\ = \sum_{m=0}^{\infty} \frac{(a, -m)(\beta, m)}{(1, m)(\gamma, -m)} x^m F_3(\beta_1, \beta+m, \alpha_1, a+m; \gamma-m; y, z)$$

using the integral (3) of [2; p. 230]

$$= \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\alpha_1)\Gamma(\gamma-\alpha-\alpha_1)} \sum_{m=0}^{\infty} \int \int u^{\alpha_1-1} v^{\alpha-1} (1-u-v)^{-\gamma-\alpha-\alpha_1-1} (1-uy)^{-\beta_1(1-vz)^{\beta}} \\ \times \frac{(\beta, m)}{(1, m)} \left\{ \frac{x}{v(1-u-v)^2(1-vz)} \right\}^m du dv$$

provided  $Re(\alpha) > 0$ ,  $Re(\alpha_1) > 0$ ,  $Re(\gamma-\alpha-\alpha_1) > 0$  and  $u \geq 0$ ,  $v \geq 0$ ,  $u+v \leq 1$ .

Now if  $|x| < r$ ,  $|y| < s$ ,  $|z| < t$ , then

$$\left| \frac{x}{v(1-u-v)^2(1-vz)} \right| < \frac{r}{1-t} \\ < 1 \text{ if } r+t < 1.$$

Therefore, the change of order of summation and integration is justified and we have

$$(2.2) \quad G_C\left(\frac{1}{\alpha}, \alpha_1, \alpha, \beta, \beta_1, \beta; \frac{1}{\gamma}, \gamma, \gamma; x, y, z\right) \\ = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\alpha_1)\Gamma(\gamma-\alpha-\alpha_1)} \int_0^1 \int_0^1 u^{\alpha_1-1} v^{\alpha-1} (1-u-v)^{\beta-\gamma-\alpha-\alpha_1-1} (1-uy)^{-\beta_1} \times \\ \times [u(1-u-v)^2(1-vz)-x]^{-\beta} du dv$$

provided  $Re(\alpha) \geq 0$ ,  $Re(\alpha_1) \geq 0$ ,  $Re(\gamma-\alpha-\alpha_1) \geq 0$ ,  $r+t < 1$  ( $|x| < r$ ,  $|y| < s$ ,  $|z| < t$ )

Similarly, we can find

$$(2.3) \quad G_D\left(\frac{1}{\alpha}, \alpha_1, \alpha, \beta_1, \beta_2, \beta_3; \frac{1}{\gamma}, \gamma, \gamma; x, y, z\right) \\ = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\alpha_1)\Gamma(\gamma-\alpha-\alpha_1)} \int \int_{\substack{u \geq 0, v \geq 0 \\ u+v \leq 1}} u^{\alpha_1-1} v^{\alpha+\beta_1-1} (1-u-v)^{-\gamma-\alpha-\alpha_1+2\beta_1-1} (1-uy)^{-\beta_2} \\ \times (1-vz)^{-\beta_3} [v(1-u-v)^2-x]^{-\beta_1} du dv$$

where  $Re(\alpha) > 0$ ,  $Re(\alpha+\beta_1) > 0$ ,  $Re(\gamma+2\beta_1-\alpha-\alpha_1) \geq 0$ , and  $|x| < 1$ .

Again from the definition of  $G_D$ ,

$$G_D \left( \frac{1}{\alpha}, \alpha_1, \alpha, \beta_1, \beta_2, \beta_3; \frac{1}{\gamma}, \gamma, \gamma; x, y, z \right) \\ \sum_{n=0}^{\infty} \frac{(\alpha_1, n) (\beta_2, n)}{(1, n) (\gamma, n)} y^n G_2(\beta_3, \beta_1, \alpha, 1 - \gamma + n; -x, -z)$$

using the integral [3]

$$G_2(\alpha, \alpha', \beta, \beta'; x, y) = \int_{-1}^{(0, \infty+)} u^{\beta-1} (u+1)^{-\beta-\beta'} \left(1 - \frac{x}{u}\right)^{-\alpha} (1-uy)^{-\alpha'} du \frac{\Gamma(1-\beta) \Gamma(1-\beta')}{2i\pi \Gamma(1-\beta-\beta')} \\ [Re(\beta+\beta') < 1]$$

we get

$$G_D = \frac{\Gamma(1-\alpha) \Gamma(\gamma)}{2i\pi \Gamma(\gamma-\alpha)} \sum_{n=0}^{\infty} \int u^{\alpha-1} (u+1)^{\gamma-\alpha-1} \left(1 + \frac{x}{u}\right)^{-\beta_3(1+uz)-\beta_1} \cdot \frac{(\alpha_1, n) (\beta_2, n)}{(1, n) (\gamma-\alpha, n)} \times \\ \times [y(1+u)]^n du$$

changing the order of integration and summation which is justified for  $|y| < 1$ , we have

$$(2.4) \quad = \frac{\Gamma(1-\alpha) \Gamma(\gamma)}{2\pi i \Gamma(\gamma-\alpha)} \int_{-1}^{(0, \infty+)} u^{\alpha-1} (u+1)^{\gamma-\alpha-1} \left(1 + \frac{x}{u}\right)^{-\beta_3(1+uz)-\beta_1} \times \\ \times {}_2F_1(u_1, \beta_2; \gamma - \alpha; (1+u)y) du$$

where  $Re(\gamma) > Re(\alpha) > 0$ .

using integral [2; p. 232], in (2.1), we get

$$G_C \left( \frac{1}{\alpha}, \alpha_1, \alpha, \beta, \beta_1, \beta; \frac{1}{\gamma}, \gamma, \gamma; x, y, z \right) \\ = \frac{\Gamma(\gamma)}{\Gamma(\alpha_1) \Gamma(\beta_1) (2\pi i)^2} \sum_{m=0}^{\infty} \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \Gamma(-s) \Gamma(-t) (-y)^s (-z)^t \frac{\Gamma(\alpha_1+s) \Gamma(\alpha+t) \Gamma(\beta_1+s)}{\Gamma(\alpha) \Gamma(\beta) \Gamma(\gamma+s+t)} \times \\ \times \frac{\Gamma(\beta+t) (1-\gamma-s-t, m) (\beta+t, m)}{(1, m) (1-\alpha-t, m)} x^m ds dt \\ (2.5) \quad = \frac{\Gamma(\gamma)}{\Gamma(\alpha) \Gamma(\alpha_1) \Gamma(\beta) \Gamma(\beta_1)} \frac{1}{(2\pi i)^2} \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \frac{\Gamma(\alpha_1+s) \Gamma(\alpha+t) \Gamma(\beta_1+s) \Gamma(\beta+t)}{\Gamma(\gamma+s+t)} \times \\ \times \Gamma(-s) \Gamma(-t) (-y)^s (-z)^t {}_2F_1(1-\gamma-s-t, \beta+t; 1-\alpha-t; x) ds dt$$

Again using the integral for  ${}_2F_1[1]$ , we get

$$(2.6) \quad \frac{\Gamma(\gamma)}{\Gamma(\alpha) \Gamma(\alpha_1) \Gamma(\beta) \Gamma(\beta_1)} \frac{1}{(2\pi i)^3} \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \frac{\Gamma(\alpha_1+s) \Gamma(\alpha+t) \Gamma(\beta_1+s) (\beta+t)}{\Gamma(\gamma+s+t)} \\ \times \frac{\Gamma(1-\alpha-t) \Gamma(1-\gamma-s-t+r) \Gamma(\beta+t+r)}{\Gamma(\beta+t) \Gamma(1-\gamma-s-t) \Gamma(1-\alpha-t+r)} (-x)^r (-y)^s (-z)^t dr ds dt.$$

Similarly, we can show that

$$(2.7) \quad G_D \left( \frac{1}{\alpha}, \alpha_1, \alpha, \beta_1, \beta_2, \beta_3; \frac{1}{\gamma}, \gamma, \gamma; x, y, z \right)$$

$$\begin{aligned}
&= \frac{\Gamma(\gamma)}{\Gamma(\alpha) \Gamma(\alpha_1) \Gamma(\beta_2) \Gamma(\beta_3)} \frac{1}{(2\pi i)^2} \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \frac{\Gamma(\alpha_1+s) \Gamma(\alpha+t) \Gamma(\beta_2+s) \Gamma(\beta_3+t)}{\Gamma(\gamma+s+t)} \times \\
&\quad \times \Gamma(-s) \Gamma(-t) (-y)^s (-z)^t {}_2F_1(1-\gamma-s-t, \beta_1; 1-\alpha-t; x) ds dt \\
(2.8) \quad &= \frac{\Gamma(\gamma)}{\Gamma(\alpha) \Gamma(\alpha_2) \Gamma(\beta_1) \Gamma(\beta_2) \Gamma(\beta_3)} \cdot
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{(2\pi i)^3} \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \frac{\Gamma(\alpha_1+s) \Gamma(\alpha+t) \Gamma(\beta_2+s) \Gamma(\beta_3+t)}{\Gamma(\gamma+s+t)} \times \\
&\times \frac{\Gamma(1-\alpha-t) \Gamma(\beta_1+r) \Gamma(1-\gamma-s-t+r)}{\Gamma(1-\gamma-s-t) \Gamma(1-\alpha-t+r)} \Gamma(-r) \Gamma(-s) \Gamma(-t) (-x)^r (-y)^s (-z)^t dr ds dt.
\end{aligned}$$

### 3. Transformations and Reducible cases

We can easily write

$$\begin{aligned}
&G_C \left( \frac{1}{\alpha}, \alpha_1, \alpha, \beta, \beta_1, \beta; \frac{1}{\gamma}, \gamma, \gamma; x, y, z \right) \\
&= \sum_{m=0}^{\infty} \frac{(\alpha, -m) (\beta, m)}{(1, m) (\gamma, -m)} x^m {}_3F_3(\alpha - m, \alpha_1, \beta + m, \beta_1; \gamma - m; y, z)
\end{aligned}$$

and using the transformation [2; p. 241]

$$F_3(\alpha, \alpha', \beta, \beta'; \alpha + \alpha'; x, y) = (1-y)^{-\beta'} F_1\left(\alpha, \beta, \beta'; \alpha + \alpha'; x, \frac{y}{y-1}\right).$$

We get the transformation

$$\begin{aligned}
(3.1) \quad &G_C \left( \frac{1}{\alpha}, \alpha_1, \alpha, \beta, \beta_1, \beta; \frac{1}{\gamma}, \gamma, \gamma; x, y, z \right) \\
&= (1-z)^{-\beta_1} G_A \left( \frac{1}{\alpha}, \alpha, \alpha, \beta, \beta_1, \beta; \frac{1}{\alpha + \alpha_1}, \alpha + \alpha_1, \alpha + \alpha_1; x, y, \frac{z}{z-1} \right)
\end{aligned}$$

Proceeding in the similar way, we can derive transformation for the function  $G_D$  as

$$\begin{aligned}
(3.2) \quad &G_D \left( \frac{1}{\alpha}, \alpha_1, \alpha, \beta_1, \beta_2, \beta_3; \frac{1}{\gamma}, \gamma, \gamma; x, y, z \right) \\
&= (1-z)^{-\beta_3} G_B \left( \frac{1}{\alpha}, \alpha, \alpha, \beta_1, \beta_2, \beta_3; \frac{1}{\alpha + \alpha_1}, \alpha + \alpha_1, \alpha + \alpha_1; x, y, \frac{z}{z-1} \right)
\end{aligned}$$

Again from the definition of  $G_D$  and the known identity [2; p. 239]

$$F_3(\alpha, \alpha', \beta, \beta'; \gamma; x, x) = {}_2F_1(\alpha, \beta + \beta'; \gamma; x)$$

we get

$$\begin{aligned}
(3.3) \quad &G_D \left( \frac{1}{\alpha}, \alpha_1, \alpha, \beta_1, \beta_2, \beta_3; \frac{1}{\gamma}, \gamma, \gamma; x, y, z \right) \\
&= (1-z)^{-\beta_3} G_2 \left( \beta_2 + \beta_3, \beta_1, \alpha_1, 1 - \alpha - \alpha_1; x, \frac{z}{z-1} \right)
\end{aligned}$$

I shall now use the integral (2.4) to find the various transformations.  
On making the following substitutions

$$(i) \quad u = -x + (1-x)v$$

$$(ii) \quad u = \frac{v}{1-x-vz}$$

in the integral (2.4), we are immediately led to the following transformations

$$(3.4) \quad G_D \left( \frac{1}{\alpha}, \alpha_1, \alpha, \beta_1, \beta_2, \beta_3; \frac{1}{\gamma}, \gamma, \gamma; x, y, z \right) \\ = (1-x)^{\gamma-1} (1-xz)^{-\beta_1} G_D \left( \frac{1}{\alpha}, \alpha_1, \alpha, \beta_1, \beta_2, \alpha+\beta_3-1; \frac{1}{\gamma}, \gamma, \gamma; \frac{-x}{1-x}, (1-x)y; \frac{(1-x)z}{1-xz} \right)$$

$$(3.5) = (1-xz)^{-\beta_3} (1-z)^{-\alpha+1} G_D \left( \frac{1}{\alpha}, \alpha_1, \alpha, \gamma-\beta_1-z, \beta_2, \beta_3; \frac{1}{\gamma}, \gamma, \gamma; \frac{x(1-z)}{1-xz} (1-z)y, \frac{-z}{1-z} \right)$$

#### 4. Expansions

We know

$$G_C \left( \frac{1}{\alpha}, \alpha_1, \alpha, \beta, \beta_1, \beta; \frac{1}{\gamma}, \gamma, \gamma; x, y, z \right) \\ = \sum_{m=0}^{\infty} \frac{(\alpha, -m) (\beta, m)}{(1, m) (\gamma, m)} x^m {}_2F_3(\alpha_1, \alpha-m, \beta_1, \beta+m; \gamma-m; y, z)$$

using the result [I].

$${}_2F_3(a_1, a_2, b_1, b_2; c; x, y) = \sum_{m=0}^{\infty} \frac{(-1)^m (a_1, m) (a_2, m) (b_1, m)}{(1, m) (c, m) (c-a_2, m)} x^m \times$$

$$\times {}_2F_1(a_1+m, b_1+m; c-a_2+m; x) {}_2F_1(a_2+m, b_2; c+m; y)$$

we get

$$(4.1) \quad G_C \left( \frac{1}{\alpha}, \alpha_1, \alpha, \beta, \beta_1, \beta; \frac{1}{\gamma}, \gamma, \gamma; x, y, z \right) \\ = \sum_{r=0}^{\infty} \frac{(-1)^r (\alpha, r) (\alpha_1, r) (\beta_1, r)}{(1, r) (\gamma, r) (\gamma-\alpha, r)} y^r \cdot {}_2F_1(\alpha_1+r, \beta_1+r; \gamma-\alpha+r; y) \times \\ \times G_1(\beta, \alpha+r, 1-\gamma-r; x, z)$$

Similarly, we can prove

$$(4.2) \quad G_D \left( \frac{1}{\alpha}, \alpha_1, \alpha, \beta_1, \beta_2, \beta_3; \frac{1}{\gamma}, \gamma, \gamma; x, y, z \right) \\ = \sum_{r=0}^{\infty} \frac{(-1)^r (\alpha, r) (\alpha_1, r) (\beta_2, r)}{(1, r) (\gamma, r) (\gamma-\alpha, r)} y^r \cdot {}_2F_1(\alpha_1+r, \beta_2+r; \gamma-\alpha+r; y) \times \\ \times G_2(\beta_2, \beta_1, \alpha+r, 1-\gamma-r; x, z)$$

## 5. System of partial differential equations

On setting  $\theta = x \frac{\partial}{\partial x}$ ,  $\phi = y \frac{\partial}{\partial y}$  and  $\psi = z \frac{\partial}{\partial z}$

it can be readily seen that  $G_C$  satisfies the differential equations

$$(5.1) \quad \begin{cases} [\theta(\alpha + \psi - \theta) - x(\beta + \theta + \psi)(\gamma + \phi + \psi - \theta - 1)] G_C = 0, \\ [\phi(\gamma + \phi + \psi - \theta - 1) - y(\alpha_1 + \phi)(\beta_1 + \phi)] G_C = 0, \\ [\psi(\gamma + \phi + \psi - \theta - 1) - z(\alpha + \psi - \theta)(\beta + \theta + \psi)] G_C = 0, \end{cases}$$

and that the equations associated with  $G_D$  are

$$(5.2) \quad \begin{cases} [\theta(\alpha + \psi - \theta) - x(\beta_1 + \theta)(\gamma + \phi + \psi - \theta)] G_D = 0, \\ [\phi(\gamma + \phi + \psi - \theta - 1) - y(\beta_2 + \phi)(\alpha_1 + \phi)] G_D = 0, \\ [\psi(\gamma + \phi + \psi - \theta - 1) - z(\beta_3 + \psi)(\alpha_1 + \psi - \theta)] G_D = 0. \end{cases}$$

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**Absorbance curves of the complexes:** A series of mixtures containing niobium or tantalum solution and ARS in different proportions (0 : 1, 2 : 1, 1 : 1, 1 : 2 and 1 : 3) were prepared and their absorbances were measured spectrophotometrically. It was found that the region of maximum absorbance in case of niobium is 470 m $\mu$  at pH 4.0 while in case of tantalum it is 460 m $\mu$  at pH 5.0. Most of the studies described were carried out at 470 and 490 m $\mu$  in both the cases.

**Effect of time and heating.** The complex formation was not found to be instantaneous, and full colour intensity develops after keeping the solutions for two hours. No change in absorbance was observed when the complex solutions were heated.

**Effect of pH:** The effect of change in pH on the  $\lambda_{max}$  of the Nb - ARS chelate and Ta - ARS chelate have been shown in Figs. 1 and 2. The  $\lambda_{max}$  remains unchanged between pH 3.5 to 5.5 in case of niobium and 4.5 to 5.5 in case of tantalum. The variation in absorbance of the Nb - ARS chelate has been studied with change in pH at 470 m $\mu$  and the maximum colour intensity is found at pH 4.0 (Fig. omitted).

**Validity of Beer's law:** At the wavelength 470 m $\mu$ , Beer's law is obeyed over the range 0.93 - 4.14 p.p.m. of niobium and 1.81 - 32.60 p.p.m. of tantalum.

**Sensitivity:** The sensitivity of the reaction as defined by Sandell<sup>8</sup> is 0.81  $\gamma/cm^2$  for niobium and 0.32  $\gamma/cm^2$  for tantalum.

**Interference due to diverse ions:** The effect of a number of diverse ions on the system was studied. The cations which interfere are: Ag(I), Tl(I), Pb(II), Hg(II), Bi(III), Cu(II), Cd(II), As(III), Sb(III), Fe(II, III), Al(III), Cr(III), Mn(II), Zn(II), Ni(II), Co(II), Ba(II), Sr(II), Ca(II), Mg(II), Be(II), Th(IV), Zr(IV), Ce(III, IV), W(VI), Mo(VI), U(VI), Ga(III), In(III), Ru(III) and Rh(III).

**Procedure for the determination:** For the determination of niobium and tantalum spectrophotometrically, the solution should be free from other interfering ions. 5-fold molar excess of ARS is introduced to the aliquot portion of the sample and the pH is adjusted to 4.0 and 5.0 respectively. The mixtures are kept for two hours for equilibration and the absorbances are measured at 470 m $\mu$ . The metal content is determined by comparing the absorbance value with a calibration curve prepared under identical conditions.

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## **Studies in the influence of Direct application of Organic matter and application of Compost obtained from the same materials on the yield of wheat crop and soil fertility with and without phosphate**

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### **Abstract**

Influence of direct application of organic materials (Wheat straw and Weed) and their compost with and without phosphates (Tata and Kulti basic slags) on the yield of Wheat crop (N. P. 710) and soil fertility were studied. It is found that the direct application of organic materials produces more yield of both grains and straw than the compost formed from the same organic matter on equivalent nitrogen basis. Wheat straw as an organic matter is found to be more useful. Oxidation of carbonaceous compounds of soil humus and those added by ploughing in of organic matter is greatly facilitated by adding basic slag and thus helps the land to become more fertile and productive. It is found that direct ploughing of organic matter is certainly better than the addition of compost obtained from the same organic matter composted in pits, because in direct ploughing of organic matter, there is much more nitrogen fixation than in composting where even marked losses of nitrogen have been found.

At the symposium on the 'Fertility of Indian soils', Raychaudhuri<sup>1</sup> stated "Essentially, however, the main cause of low fertility status of Indian soils is its low contents of soil organic matter". Dutta and Goswami<sup>2</sup> reported: "The importance of organic matter in soil and fertility is well recognised and the building up and maintenance of organic matter in the soil are, therefore, regarded, amongst other things, primary objective of good soil management."

In view of the vital role of organic matter in maintaining soil fertility and its general deficiency in our soils, one of the major problem of Indian agriculture is to build up soil organic matter and to conserve it. There are evidences based on experimental findings to suggest that organic matter content of Indian soils can be built up. Mariakulandai<sup>3</sup> stated that green matter at 5000 lbs./acre on sandy loam soils improved the fertility status of soils with regard to organic matter and nitrogen content. Dhar<sup>4</sup> first emphasized that the direct application of plant materials to the field, before composting, is more beneficial to crops because the energy materials like carbohydrates, lignin, fats etc. when added to the soil, are partially oxidised and in this process, helping to get nitrogen of the air fixed. Hence, much more humus is formed and added to the soil when plant materials are directly mixed with soil instead of their application after composting them elsewhere. Shivaraman<sup>5</sup> stated: "Addition of crop residues has been considered to be the best way of fertility management of tropical soils."



## Experimental

In the light of the above facts, we have carried on comparative Pot experiments on the influence of different organic materials and basic slags directly applied to soil and applied after being composted with and without phosphates.

For these experiments, the soil was collected from the University Farm. 5 kgs lots of clean and powdered soil were taken in each Pot. To this organic materials viz., Wheat straw and Weed (Powdered and sieved through 80 mesh) were added at the rate of 10 tons/acre. Phosphates were added to the mixture of soil + organic matter at the rate of 50 lbs. per acre in the form of Tata and Kulti basic slags. Composts from these organic materials were already prepared with 0.5%  $P_2O_5$  from basic slags and without phosphate. To weighed quantity of 5 kgs. of soil, different composts were added to make up 100 lbs. N per acre. Four replications were carried out for each treatment.

Soil + straw + phosphates were thoroughly mixed to make the mixture uniform and thereafter the mixture was filled up in the earthen Pots. The pots were exposed to the sunlight and the contents were thoroughly stirred. One month time was given to allow the process of oxidation of organic matter to set in, after which the wheat crop (N. P. 710) was sown. In case of composts and soil, the mixture was thoroughly mixed in and filled up in the pots. 20% water by weight, was added to keep the system moist. After 15 days of exposure and two stirrings, the crop was sown.

### *Analysis of Soil and different Organic materials*

Estimation %	Soil	Wheat straw	Weed
Loss on Ignition	4.2153	91.1230	90.1230
		Ash—	
HCl insoluble	85.6235	8.7645	9.5623
Sesquioxide	10.0529	5.1240	5.8234
$Fe_2O_3$	3.3212	1.3460	0.7790
Total CaO	0.9872	0.6130	0.3850
Total MgO	0.5254	0.7806	0.7540
Total $P_2O_5$	0.0945	0.3482	0.5545
Available $P_2O_5$	0.0156	0.5140	0.5786
Total $K_2O$	0.9725	0.6655	0.7340
Total carbon	0.5720	38.6245	40.2456
Total nitrogen	0.0561	0.6390	0.6898
$NH_3-N$	0.0052		
$NO_3-N$	0.0082		
C/N ratio	10.19	60.44	58.34

### *Analysis of Basic (Thomas) slag*

Estimation	Tata basic slag	Kulti basic slag
$SiO_2$	16.3212	16.5432
$Fe_2O_3$	12.8265	10.8000
CaO	37.9684	37.2135
MnO	2.9076	2.9978
MgO	4.9736	4.4280
$K_2O$	0.6481	0.3364
Total $P_2O_5$	7.0230	4.1680
Available $P_2O_5$ (2% Citric acid)	4.1021	2.0340

*Analysis of different Composts prepared from different Organic materials with and without phosphates*

Wheat straw compost				Weed composts			
Phosphated		Unphosphated		Phosphated		Unphosphated	
T. B. S.	K. B. S.	...	...	T. B. S.	K. B. S.	...	...
10.90	11.60	Total carbon	24.00	18.00	15.00	Total carbon	24.80
1.55	1.51	Total Nitrogen	1.13	1.58	1.53	Total Nitrogen	1.15
0.0832	0.0782	NH <sub>3</sub> -N	0.0532	0.0437	0.0412	NH <sub>3</sub> -N	0.0452
0.1600	0.1466	NO <sub>3</sub> -N	0.1025	0.1015	0.0996	NO <sub>3</sub> -N	0.0988
0.4014	0.3625	Available P <sub>2</sub> O <sub>5</sub>	0.2722	0.3712	0.3610	Available P <sub>2</sub> O <sub>5</sub>	0.2715

TABLE 1

*Showing the yield of Wheat grains and Wheat straw/5 plants in gms. as influenced by Direct addition of Organic matter with and without phosphates*

Treatments	Yield of Wheat grains		Yield of Wheat straw	
	Average	Standard Error	Average	Standard Error
Soil alone	56.4	±1.8814	104.8	±0.6422
Soil + wheat straw	153.2	±1.1640	274.2	±0.7648
Soil + Wheat straw + T.B.S.	173.2	±0.5049	290.3	±2.9127
Soil + Wheat straw + K.B.S.	162.5	±1.4168	278.0	±1.2250
Soil + Weed	140.0	±1.1149	242.0	±3.509
Soil + Weed + T.B.S.	166.6	±2.3731	273.7	±1.1874
Soil + K.B.S.	160.0	±1.9141	258.0	±1.9249

TABLE 2

*Showing the yield of Wheat grains and Wheat straw/5 plants in gms. as influenced by different composts, composted with and without Phosphates*

Treatments	Yield of Wheat grains		Yield of What straw	
	Average	Standard Error	Average	Standard Error
Soil alone	54.6	±2.099	93.5	±1.1099
Soil + Wheat straw	62.0	±1.3285	119.3	±1.4370
Soil + Wheat straw + T.B.S.	66.5	±0.4123	124.2	±1.2509
Soil + Wheat straw + K.B.S.	62.6	±2.0999	122.6	±1.8909
Soil + Weed	57.2	±0.6480	117.5	±1.10000
Soil + Weed + T.B.S.	64.0	±0.3900	120.3	±0.8972
Soil + Weed + K.B.S.	58.2	±0.6745	119.3	±1.1489

TABLE 3

(By direct application of Organic matter)

*Showing the amount of Nitrogen initially present, amount left in the soil after harvest, recovery of Nitrogen and the amount gained or lost in the Pots.*

Treatments	Nitrogen initially present at sowing time (gms.)	Nitrogen left in soil after harvest (gms.)	Nitrogen recovered in grains (gms.)	Nitrogen recovered in straw (gms.)	Total recovery (gms.)	Nitrogen gained or lost (gms.)
Soil alone	2·8000	1·1000	0·85	0·65	1·50	-0·2000
Soil + Wheat straw	3·0898	2·3000	2·10	1·70	3·80	+3·0102
Soil + Wheat straw + T.B.S.	3·0898	2·4000	2·50	1·80	4·30	+3·6102
Soil + Wheat straw + K.B.S.	3·0898	2·1000	2·40	1·70	4·10	+3·1102
Soil + Weed	3·1128	2·2000	2·30	1·50	3·80	+2·8872
Soil + Weed + T. B. S.	3·1128	2·4000	2·60	1·70	4·30	+3·5872
Soil + Weed + K. B. S.	3·1128	2·2000	2·40	1·60	4·00	+3·0872

TABLE 4

(Addition of organic matter after composting)

*Showing the amount of Nitrogen initially present, amount left in the soil after harvest, recovery of Nitrogen and the amount gained or lost in the Pots.*

Treatments	Nitrogen initially present at sowing time (gms.)	Nitrogen left in soil after harvest (gms.)	Nitrogen recovered in grains (gms.)	Nitrogen recovered in straw (gms.)	Total recovery (gms.)	Nitrogen gained or lost (gms.)
Soil alone	2·8000	1·0000	0·8200	0·5800	1·40	-0·4000
Soil + Wheat straw	3·0260	1·0600	0·9300	0·7700	1·70	-0·2660
Soil + Wheat straw + T.B.S.	3·0260	1·2000	0·9400	0·7600	1·70	-0·1260
Soil + Wheat straw + K.B.S.	3·0260	1·1000	0·9600	0·7400	1·70	-0·2260
Soil + Weed	3·0260	0·9400	0·8600	0·7400	1·60	-0·4860
Soil + Weed + T.B.S.	3·0260	1·1000	0·9600	0·7400	1·70	-0·2260
Soil + Weed + K.B.S.	3·0260	1·1000	0·8600	0·7400	1·60	-0·3260

## Discussion

From the foregoing results, it is evident that direct ploughing in of organic matter increases yield of both Wheat grain and straw than that from the compost prepared from the same organic matter. Moreover, when basic slags were incorporated with these organic materials or composts, the yield of Wheat grain and straw markedly increased over the unphosphated treatments. Basic slag being alkaline due to its contents of  $\text{CaO}$ ,  $\text{K}_2\text{O}$ ,  $\text{MgO}$  etc., the rate of oxidation of cellulose and lignin in straw is hastened when it is mixed with straw and ploughed in the soil. In this process of oxidation energy is liberated which is utilised in fixing nitrogen on the soil surface and land fertility is increased. Oxidation of carbonaceous compounds of organic matter, in compost and straw added, is greatly facilitated in the presence of basic slag. In this way not only the total nitrogen of the system is increased by ploughing in basic slag and straw but, due to increased oxidation in the soil by adding basic slag, available nitrogen, phosphate, potash, lime, etc. making the land more fertile and productive. It is found that direct ploughing in of organic matter is certainly better than the addition of composts obtained from the same organic matter. Because in the direct ploughing in of the organic matter, there is much more nitrogen fixation than in composting where even marked losses of nitrogen have been found and also reported by other soil scientists, and the recovery of added nitrogen in the form of composts is not so efficient. That direct ploughing in of organic matter helps to fix atmospheric nitrogen. There is loss of nitrogen, when compost is applied after it is prepared from straw.

Dhar<sup>6</sup> observed that a mixture of organic substances like straw, plant residue etc. and calcium phosphate when incorporated into soil, can build up the soil fertility permanently by fixing atmospheric nitrogen and making available nitrogen phosphate, potash, trace elements and humus, and thus maintain soil fertility. Lady Balfour<sup>7</sup> reported in her experiments that the plots to which basic slag was added to the straw contained the largest amount of nitrogen and produced the biggest crop of barley. Bjalfve<sup>8</sup> has also observed marked nitrogen fixation more in light than that in the dark by incorporating straw in soils or sand and reported that calcium phosphates largely increase this nitrogen fixation in the soil.

From the results of these pot experiments, we have come to the conclusion that a large amount of nitrogen fixation and fertility improvement on a world wide scale are possible by ploughing in different organic materials directly to the soil in combination with phosphates. These results point out a practical way of building up the soil fertility on a permanent basis.

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# An inversion formula for Fox-H-Transform

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## Abstract

Very recently Verma introduced Fox-H-Transform by means of the integral equation

$$\phi(u) = \int_0^\infty H(u x) f(x) dx,$$

where  $H(x)$  is the H-Function of Fox defined as

$$H(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\prod_{m=1}^q \Gamma(b_m + c_m.s) \prod_{n=1}^p \Gamma(a_n - e_n.s)}{\prod_{m=1}^q \Gamma(b_m + c_m - c_m.s) \prod_{n=1}^p \Gamma(a_n - e_n + e_n.s)} x^{-s} ds.$$

Using the notation due to Gupta we write

$$\phi(u) = \lambda u \int_0^\infty H_{p,q}^{m,n} \left[ c u x \left| \begin{matrix} (a_p, e_p) \\ (b_q, f_q) \end{matrix} \right. \right] f(x) dx$$

and

$$H_{p,q}^{m,n} \left[ x \left| \begin{matrix} (a_p, e_p) \\ (b_q, f_q) \end{matrix} \right. \right] = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\prod_{j=1}^m \Gamma(b_j - f_j.s) \prod_{j=1}^n \Gamma(1 - a_j + e_j.s)}{\prod_{j=m+1}^q \Gamma(1 - b_j + f_j.s) \prod_{j=n+1}^p \Gamma(a_j - e_j.s)} x^s ds.$$

In this paper we find an inversion formula for this transform. Since Fourier sine-; cosine-; Hankel-;  $K$ -;  $\mathcal{L}$ -;  $H$ -; and Generalized Hankel-Transforms are particular cases of this transform, the inversion formula gives, on specializing the parameters, the corresponding inversion formulae of the said transforms.

In addition to the inversion formula we give here a general relation involving this transform and deduce two results of Sharma therefrom.

## 1. Introduction

In a very recent paper Verma[8] has introduced Fox-H-Transform as

$$(1.1) \quad \phi(u) = \int_0^\infty H(ux) f(x) dx$$

where

$$(1.2) \quad H(x) = \frac{1}{2\pi i} \int_{k_1 - i\infty}^{k_1 + i\infty} \frac{\prod_{m=1}^q \Gamma(b_m + c_m.s) \prod_{n=1}^p \Gamma(a_n - e_n.s) x^{-s}}{\prod_{m=1}^q \Gamma(b_m + c_m - c_m.s) \prod_{n=1}^p \Gamma(a_n - e_n + e_n.s)} ds$$

is the H-Function of Fox [4, p. 408(52)]

Following Gupta [5, p. 98(4)] we write (1.2) in the form

$$(1.3) \quad H_{p,q}^{m,n} \left[ x \left| \begin{matrix} (a_p, e_p) \\ (b_q, f_q) \end{matrix} \right. \right] = \frac{1}{2\pi i} \int_{k_1 - i\infty}^{k_1 + i\infty} \frac{\prod_{j=1}^m \Gamma(b_j - f_j.s) \prod_{j=1}^n \Gamma(1 - a_j + e_j.s) x^s}{\prod_{j=m+1}^q \Gamma(1 - b_j + f_j.s) \prod_{j=n+1}^p \Gamma(a_j - e_j.s)} ds$$

and put (1.1) in the form

$$(1.4) \quad \phi(u) = \lambda u \int_0^\infty H_{p,q}^{m,n} \left[ c u x \left| \begin{matrix} (a_p, e_p) \\ (b_q, f_q) \end{matrix} \right. \right] f(x) dx$$

where in (1.3) empty product is interpreted as unity;  $p, q, m$  and  $n$  are integers satisfying  $0 \leq n \leq p$ ,  $1 \leq m \leq q$ ;  $e_j$ , ( $j = 1, 2, \dots, p$ );  $f_j$ , ( $j = 1, 2, \dots, q$ ) are positive numbers and  $a_j$ , ( $j = 1, 2, \dots, p$ );  $b_j$ , ( $j = 1, 2, \dots, q$ ) are complex numbers such that no pole of  $\Gamma(b_h - f_h.s)$ , ( $h = 1, 2, \dots, m$ ) coincides with any pole of  $\Gamma(1 - a_i + e_i.s)$ , ( $i = 1, 2, \dots, n$ ), that is,

$$(1.5) \quad e_i(b_h + \nu) \neq (a_i - \sigma - 1)f_h \\ (\nu, \sigma = 0, 1, 2, \dots; h = 1, 2, \dots, m; i = 1, 2, \dots, n)$$

The contour runs from  $k_1 - i\infty$  to  $k_1 + i\infty$  such that the poles of  $\Gamma(b_h - f_h.s)$ , ( $h = 1, 2, \dots, m$ ) lie on the right and the poles of  $\Gamma(1 - a_i + e_i.s)$ , ( $i = 1, 2, \dots, n$ ) lie on the left of the contour. Such a contour exists on account of (1.5).

According to Braaksma [2, p. 278]

$$(1.6) \quad H_{p,q}^{m,n} \left[ x \left| \begin{matrix} (a_p, e_p) \\ (b_q, f_q) \end{matrix} \right. \right] = O(|x|^{\sigma_1}) \text{ for large } x$$

where  $\sum_{j=1}^p e_j - \sum_{j=1}^q f_j < 0$ ;  $\sum_{j=1}^n e_j - \sum_{j=n+1}^p e_j + \sum_{j=1}^m f_j - \sum_{j=m+1}^q f_j > 0$ .

$$\sigma_1 = \max R \left( \frac{a_i - 1}{e_i} \right), (i = 1, 2, \dots, n)$$

and

$$(1.7) \quad H_{p,q}^{m,n} \left[ x \left| \begin{matrix} (a_p, e_p) \\ (b_q, f_q) \end{matrix} \right. \right] = O(|x|^{\sigma_2}) \text{ for small } x,$$

where  $\sum_1^p e_j - \sum_1^q f_j \leq 0$ , and  $\sigma_2 = \min R \left( \frac{b_h}{f_h} \right) (h = 1, 2, \dots, m)$

The object of this paper is to invert the integral (1.4) and to obtain a solution which may serve as a key formula from which a number of inversion formulae for various transform pairs may be deduced as particular cases, by specializing the parameters involved.

Conditions for validity of each transform-pair may, however, be decided individually.

Besides the inversion formula we give here a general relation involving this transform and deduce two results of Sharma [7] therefrom.

## 2. The main result

*Theorem 1.* If  $x^{k-1} \cdot f(x)$  and  $x^{-k-1} \cdot \phi(x)$  belong to  $L(0, \infty)$  where  $\phi(u)$  is given by (1.4), then

$$(2.1) \quad f(x) = \frac{c}{\lambda} \int_0^\infty \frac{\phi(u)}{u} H_{p,q}^{q-m, p-n} \left[ c u x \left| \begin{matrix} (1-a_{n+1}-e_{n+1}, e_{n+1}), \dots, (1-a_p-e_p, e_p), \\ (1-a_1-e_1, e_1), \dots, (1-a_n-e_n, e_n), \\ (1-b_{m+1}-f_{m+1}, f_{m+1}), \dots, (1-b_q-f_q, f_q), \\ (1-b_1-f_1, f_1), \dots, (1-b_m-f_m, f_m). \end{matrix} \right. \right] du$$

provided

$$(i) \quad 0 \leq n \leq p, 1 \leq m \leq q, \sum_1^p e_j - \sum_1^q f_j < 0,$$

$$(ii) \quad f(x) \text{ is continuous at } x = u, u > 0, s = k + i t, 0 < t < \infty,$$

$$(iii) \quad a + \sigma_2 + 1 > 0, b + \sigma_1 + 1 < 0, \text{ where}$$

$$f(x) = \begin{cases} O(|x|^a) \text{ for small } x, \\ O(|x|^b) \text{ for large } x, \end{cases}$$

and  $\sigma_1, \sigma_2$  are defined as in (1.6), (1.7),

$$(iv) \quad 1 + \sigma_1 < k < 1 + \sigma_2,$$

$$(v) \quad \text{the H-Function and the integral in (2.1) exist.}$$

To prove the theorems of the paper we make use of the following results :

$$(2.2) \quad \int_0^\infty H_{p,q}^{m,n} \left[ \eta x \left| \begin{matrix} (a_p, e_p) \\ (b_q, f_q) \end{matrix} \right. \right] x^{s-1} dx = \frac{\prod_{j=1}^m \Gamma(b_j + f_j \cdot s) \prod_{j=1}^n \Gamma(1 - a_j - e_j \cdot s)}{\prod_{j=m+1}^q \Gamma(1 - b_j - f_j \cdot s) \prod_{j=n+1}^p \Gamma(a_j + e_j \cdot s)} \eta^{-s}.$$

$$(2.3) \quad \int_0^\infty H_{p,q}^{m,n} \left[ \eta x \mid \begin{matrix} (a_p, e_p) \\ (b_q, f_q) \end{matrix} \right] H_{\gamma,\delta}^{\alpha,\beta} \left[ \mu x \mid \begin{matrix} (\mu_\gamma, c_\gamma) \\ (\lambda_\delta, d_\delta) \end{matrix} \right] dx$$

$$= \frac{1}{\mu} H_{\delta+p, \gamma+q}^{\beta+m, \alpha+n} \left[ \frac{\eta}{\mu} \mid \begin{matrix} (a_1, e_1), \dots, (a_n, e_n), (1-\lambda_\delta-d_\delta, d_\delta), (a_{n+1}, e_{n+1}), \dots, (a_p, e_p) \\ (b_1, f_1), \dots, (b_m, f_m), (1-\mu_\gamma-c_\gamma, c_\gamma), (b_{m+1}, f_{m+1}), \dots, (b_q, f_q) \end{matrix} \right]$$

where

$$(2.4) \quad \left[ \begin{array}{l} -\min_{1 \leq j \leq m} R\left(\frac{b_j}{f_j}\right) < R(s) < \left[ \frac{1 - \max_{1 \leq j \leq n} R(a_j)}{e_j} \right] \\ \rho = \sum_{j=1}^n e_j - \sum_{j=n+1}^p e_j + \sum_{j=1}^m f_j - \sum_{j=m+1}^q f_j > 0, |\arg(\eta)| < \frac{1}{2} \rho \pi \\ \text{with similar conditions on } \mu's, c's, \lambda's \text{ and } d's, \text{ and} \\ \min_{1 \leq j \leq n} R\left(\frac{\lambda_j}{d_j}\right) + \min_{1 \leq j \leq m} R\left(\frac{b_j}{f_j}\right) + 1 > 0, \\ \left[ \frac{1 - \max_{1 \leq j \leq n} R(a_j)}{e_j} \right] + \left[ \frac{1 - \max_{1 \leq j \leq m} R(\mu_j)}{c_j} \right] > 1. \end{array} \right]$$

The result (2.2) can be read off the definition (1.3) and (2.3) is a particular case of a result due to the author [6].

Multiplying both sides of (1.4) by  $u^{-s}$  and integrating with respect to  $u$  between 0 and  $\infty$ , we get with the help of (2.2),

$$\int_0^\infty f(x) x^{s-1} dx = \frac{c^{1-s}}{\lambda} \frac{\prod_{j=m+1}^p \Gamma\{1 - b_j + f_j(s-1)\} \prod_{j=n+1}^q \Gamma\{a_j - e_j(s-1)\}}{\prod_{j=1}^m \Gamma\{b_j - f_j(s-1)\} \prod_{j=1}^n \Gamma\{1 - a_j + e_j(s-1)\}} \int_0^\infty \frac{\phi(u) du}{u^{s+1}}$$

Therefore using Inversion Theorem for Mellin Transforms and interpreting with the help of (1.3) we get (2.1) under the conditions stated.

*Particular Cases :*

I. Putting  $\lambda = \frac{1}{\sqrt{2}}$ ,  $m=1$ ,  $n=0=p$ ,  $q=2$ ,  $c=\frac{1}{2}$ ,

$b_1 = \frac{1}{2} + \frac{\nu}{2}$ ,  $b_2 = \frac{1}{2} - \frac{\nu}{2}$ ,  $f_1 = \frac{1}{2} = f_2$ , we get

$$\frac{\phi(u)}{u} = \frac{1}{\sqrt{2}} \int_0^\infty H_{0,2}^{1,0} \left[ \frac{ux}{2} \mid \left( \frac{1}{2} + \frac{\nu}{2}, \frac{1}{2} \right), \left( \frac{1}{2} - \frac{\nu}{2}, \frac{1}{2} \right) \right] f(x) dx$$

$$= \int_0^\infty (xu)^{\frac{1}{2}} J_\nu(ux) f(x) dx.$$



and

$$f(x) = \frac{1}{\sqrt{2}} \int_0^\infty \frac{\phi(u)}{u} H_{0,2}^{1,0} \left[ \frac{ux}{2} \middle| \left( \frac{1}{4} + \frac{\nu}{2}, \frac{1}{2} \right), \left( \frac{1}{4} - \frac{\nu}{2}, \frac{1}{2} \right) \right] du$$

$$= \int_0^\infty (ux)^{\frac{1}{2}} J_\nu(ux) \cdot \frac{\phi(u)}{u} du$$

provided  $a + \frac{3}{2} + R(\nu) > 0$ ,  $b + 1 < 0$ ,

$\int_0^\infty \phi(u) \cdot u^{-s-1} du$  and  $\int_0^\infty x^{s-1} f(x) dx$  converge absolutely,  $f(x)$  is continuous at  $x = u$ ,  $u > 0$ ,  $s = k + it$ ,  $0 < t < \infty$ ,  $1 < k < \frac{3}{2} + R(\nu)$

This agrees with the inversion formulae of the Hankel-; Fourier sine-and Fourier cosine-Transforms [3, pp. 71-72].

II. Putting  $\lambda = \frac{1}{\sqrt{2}}$ ,  $m = 2$ ,  $n = 0$ ,  $p = 1$ ,  $q = 3$ ,  $c = \frac{1}{2}$ ,

$$a_1 = -\frac{\nu}{2} - \frac{1}{4}, e_1 = \frac{1}{2}, b_1 = -\frac{\nu}{2} + \frac{1}{4}, b_2 = \frac{1}{4} + \frac{\nu}{2},$$

$$b_3 = -\frac{\nu}{2} - \frac{1}{4}, f_1 = f_2 = f_3 = \frac{1}{2}, \text{ we get}$$

$$\frac{\phi(u)}{u} = \frac{1}{\sqrt{2}} \int_0^\infty H_{1,3}^{2,0} \left[ \frac{ux}{2} \middle| \begin{matrix} (-\frac{\nu}{2} - \frac{1}{4}, \frac{1}{2}) \\ (\frac{1}{4} - \frac{\nu}{2}, \frac{1}{2}), (\frac{1}{4} + \frac{\nu}{2}, \frac{1}{2}), (-\frac{1}{4} - \frac{\nu}{2}, \frac{1}{2}) \end{matrix} \right] f(x) dx$$

$$= \int_0^\infty (\lambda u)^{\frac{1}{2}} 1/\nu (xu) f(x) dx$$

and

$$f(x) = \frac{1}{\sqrt{2}} \int_0^\infty \frac{\phi(u)}{u} H_{1,3}^{1,1} \left[ \frac{ux}{2} \middle| \begin{matrix} (\frac{3}{4} + \frac{\nu}{2}, \frac{1}{2}) \\ (\frac{3}{4} + \frac{\nu}{2}, \frac{1}{2}), (\frac{1}{4} + \frac{\nu}{2}, \frac{1}{2}), (\frac{1}{4} - \frac{\nu}{2}, \frac{1}{2}) \end{matrix} \right] du$$

$$= \int_0^\infty \frac{\phi(u)}{u} (ux)^{\frac{1}{2}} H_\nu(ux) du$$

provided  $a + \frac{3}{2} \pm R(\nu) > 0$ ,  $b + 1 < 0$ ,  $\int_0^\infty \phi(u) \cdot u^{-s-1} du$

and  $\int_0^\infty f(x) x^{s-1} dx$  converge absolutely,  $f(x)$  is continuous

at  $x = u$ ,  $u > 0$ ,  $s = k + it$ ,  $0 < t < \infty$ , and  $1 < k < \frac{3}{2} \pm R(\nu)$ .

Similarly we can show that if  $\frac{\phi(u)}{u}$  is the H-Transform of  $f(x)$ , then  $f(x)$  is the Y-Transform of  $\frac{\phi(u)}{u}$ . This agrees with the inversion formulae of Y-and H-Transform pairs.

III. Putting  $\lambda = \frac{1}{2\sqrt{\pi}}$ ,  $c = \frac{1}{2} = f_1 = f_2$ ,  $n = 0 = p, q = m = 2$ ,

$$b_1 = \frac{1}{4} + \frac{\nu}{2}, b_2 = \frac{1}{4} - \frac{\nu}{2}, \text{ we obtain}$$

$$\begin{aligned}\frac{\phi(u)}{u} &= \frac{1}{2\sqrt{\pi}} \int_0^\infty H_{0,2}^{2,0} \left[ \frac{ux}{2} \middle| \left( \frac{1}{4} + \frac{\nu}{2}, \frac{1}{2} \right), \left( \frac{1}{4} - \frac{\nu}{2}, \frac{1}{2} \right) \right] f(x) dx \\ &= \sqrt{\frac{2}{\pi}} \int_0^\infty (ux)^{\frac{1}{2}} K_\nu(ux) f(x) dx\end{aligned}$$

so that

$$\begin{aligned}f(x) &= \sqrt{\pi} \int_0^\infty \frac{\phi(u)}{u} H_{0,2}^{0,0} \left[ \frac{ux}{2} \middle| \left( \frac{1}{4} + \frac{\nu}{2}, \frac{1}{2} \right), \left( \frac{1}{4} - \frac{\nu}{2}, \frac{1}{2} \right) \right] du \\ &= \frac{1}{2\pi i} \int_0^\infty \frac{\phi(u)}{u} \Gamma\left(\frac{1}{2}\right) du \int_{k-i\infty}^{k+i\infty} \left(\frac{ux}{2}\right)^{-s} ds \\ &= \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4} + \frac{\nu}{2} - \frac{s}{2}\right) \Gamma\left(\frac{3}{4} - \frac{\nu}{2} - \frac{s}{2}\right)} \int_0^\infty \phi(u) \cdot u^{-s-1} du ds\end{aligned}$$

provided  $a + \frac{s}{2} \pm R(\nu) > 0$ ,  $\int_0^\infty \phi(u) \cdot u^{-s-1} du$  and

$$\int_0^\infty x^{s-1} f(x) dx \text{ converge absolutely, } f(x) \text{ is continuous at}$$

$$x = u, u > 0, s = k + it, 0 < t < \infty, k < \frac{s}{2} \pm R(\nu).$$

This is Sharma's result [7, p. 736(2.1)]

IV. Putting  $\lambda = \frac{1}{\sqrt{2}}$ ,  $m = 1$ ,  $n = 0 = p$ ,  $q = 2$ ,  $c = \frac{1}{2}$ ,  $f_1 = \frac{1}{2}$ ,

$$f_2 = \frac{\mu}{2}, b_1 = \frac{\nu}{2} + \frac{1}{4}, b_2 = \frac{\mu\nu}{2} - \nu + \frac{\mu}{4}.$$

we get

$$\begin{aligned}\frac{\phi(u)}{u} &= \frac{1}{\sqrt{2}} \int_0^\infty H_{0,2}^{1,0} \left[ \frac{ux}{2} \middle| \left( \frac{\nu}{2} + \frac{1}{4}, \frac{1}{2} \right), \left( \frac{\nu\mu}{2} - \nu + \frac{\mu}{4}, \frac{\mu}{2} \right) \right] f(x) dx \\ &= \frac{1}{2^\nu} \int_0^\infty (xu)^{\nu+\frac{1}{2}} J_\nu\left(\frac{x^2 u^2}{4}\right) f(x) dx\end{aligned}$$

and

$$\begin{aligned}f(x) &= \frac{1}{\sqrt{2}} \int_0^\infty \frac{\phi(u)}{u} H_{0,2}^{1,0} \left[ \frac{ux}{2} \middle| \left( 1 - \frac{\nu\mu}{2} + \nu - \frac{3\mu}{4}, \frac{\mu}{2} \right), \left( \frac{1}{4} - \frac{\nu}{2}, \frac{1}{2} \right) \right] du \\ &= \frac{2 + \nu - \frac{2}{\mu}(1 + \nu)}{\mu} \int_0^\infty (ux)^{\frac{2}{\mu}(1 + \nu) - \nu - \frac{s}{2}} J_{\frac{1 + \nu - \mu}{\mu}} \left\{ \left( \frac{x^2 u^2}{4} \right)^{\frac{1}{\mu}} \right\} \frac{\phi(u)}{u} du\end{aligned}$$

which agrees with the result given by Agrawal [1, p. 69].

3. Theorem 2: If

$$(i) a + \sigma_2 + 1 > 0, b + \sigma_1 + 1 < 0,$$

$$(ii) a' + \sigma_2' > 0, b' + \sigma_1' < 0,$$

(iii) conditions (2.4) hold,

and (iv)  $\phi(u)$  is given by (1.4),

then

$$(3.1) \quad \frac{At}{\lambda} \int_0^\infty \frac{\phi(u)}{u} H_{\gamma, \delta}^{\alpha, \beta} \left[ Aut \left| \begin{matrix} (u_\gamma, c_\gamma) \\ (\lambda_\delta, d_\delta) \end{matrix} \right. \right] du \\ = \int_0^\infty f(x) H_{p+\delta, q+\gamma}^{m+\beta, n+\alpha} \left[ \frac{cx}{At} \left| \begin{matrix} (a_1, e_1), \dots, (a_n, e_n), (1-\lambda_\delta-d_\delta, d_\delta), \\ (a_{n+1}, e_{n+1}), \dots, (a_p, e_p), \\ (b_1, f_1), \dots, (b_m, f_m), (1-\mu_\gamma, c_\gamma, c_\gamma), \\ (b_{m+1}, f_{m+1}), \dots, (b_q, f_q) \end{matrix} \right. \right] dx$$

provided the H-function and the integral on the right of (3.1) exist and

$$\phi(u) = \begin{cases} O(|u|^{a'}) & \text{for small } u \\ O(|u|^{b'}) & \text{for large } u \end{cases}$$

$$\sigma_1' = \max R \left( \frac{\mu_i - 1}{c_i} \right), i = 1, 2, \dots, \beta$$

$$\sigma_2' = \min R \left( \frac{\lambda_i}{d_i} \right), i = 1, 2, \dots, \alpha.$$

This may be easily obtained by multiplying both sides of (1.4)

by  $H_{\gamma, \delta}^{\alpha, \beta} \left[ Aut \left| \begin{matrix} (\mu_\gamma, c_\gamma) \\ (\lambda_\delta, d_\delta) \end{matrix} \right. \right]$ , integrating with respect to  $u$  between 0 and  $\infty$ , and applying (2.3).

Particular cases :

Case I Putting  $\lambda = \frac{1}{2\sqrt{\pi}}, c = \frac{1}{2}, m = 2 = q, n = 0 = p,$

$$f_1 = \frac{1}{2} = f_2, b_1 = \frac{1}{4} + \frac{\nu}{2}, b_2 = \frac{1}{4} - \frac{\nu}{2}, \alpha = \beta = 1,$$

$$\gamma = 1, \delta = 3, A = \frac{1}{2}, \mu_1 = \frac{3}{4} - \frac{\nu}{2}, \lambda_1 = \frac{3}{4} - \frac{\nu}{2},$$

$$\lambda_2 = \frac{1}{4} + \frac{\nu}{2}, \lambda_3 = \frac{1}{4} - \frac{\nu}{2}, d_1 = d_2 = d_3 = \frac{1}{2} = c_1,$$

we get from (1.4) and (3.1) respectively

$$\frac{\phi(u)}{u} = \sqrt{\frac{2}{\pi}} \int_0^\infty (ux)^{\frac{1}{2}} K_\nu(ux) f(x) dx$$

and

$$\sqrt{2\pi} \int_0^\infty \frac{\phi(u)}{u} (ut)^{\frac{1}{2}} H_{-1}^\nu(ut) du \\ = \frac{1}{t} \int_0^\infty f(x) H_{1,1}^{1,1} \left[ \frac{x}{t} \left| \begin{matrix} (\frac{\nu}{2} - \frac{1}{4}, \frac{1}{2}) \\ (\frac{\nu}{2} - \frac{1}{4}, \frac{1}{2}) \end{matrix} \right. \right] dx$$

Putting

$$\psi(\nu, t) = \sqrt{\frac{\pi}{2}} \int_0^\infty \frac{\phi(u)}{u} (ut)^{\frac{1}{2}} H_{-1}^\nu(ut) du$$

and simplifying we see that

$$t^{\nu-\frac{3}{2}} \cdot \psi(\nu, t) = \int_0^\infty x^{\nu-1} (x^2+t^2)^{-1} x^{\frac{1}{2}} f(x) dx$$

Applying Mellin's Inversion Theorem we obtain under appropriate conditions

$$f(x) = \frac{x^2+t^2}{2\pi i} \int_{k'-i\infty}^{k'+i\infty} x^{-\nu-\frac{1}{2}} \cdot t^{\nu-3/2} \psi(\nu, t) d\nu.$$

which is given by Sharma [7, p. 738(4.1)].

Case II. Putting  $\lambda = \frac{1}{2\sqrt{\pi}}$ ,  $c = \frac{1}{2} = f_1 = f_2 = A$ ,  $m = 2 = q$ ,

$$\begin{aligned} n &= 0 = p, \quad b_1 = \frac{1}{4} + \frac{\nu}{2}, \quad b_2 = \frac{1}{4} - \frac{\nu}{2}, \\ a &= 1, \quad \beta = 0 = \gamma, \quad \delta = 2, \quad \lambda_1 = \frac{1}{4} + \mu + \frac{\nu}{2}, \\ \lambda_2 &= \frac{1}{4} - \frac{\nu}{2}, \quad d_1 = \frac{1}{2} = d_2, \end{aligned}$$

we get

$$\frac{\phi(u)}{u} = \sqrt{\frac{2}{\pi}} \int_0^\infty (ux)^{\frac{1}{2}} K_\nu(ux) f(x) dx$$

and

$$\begin{aligned} \frac{t^{\mu+1} \pi^{\frac{1}{2}}}{2^{\mu-\frac{1}{2}}} \int_0^\infty (ut)^{\frac{1}{2}} J_{\nu+\mu}(ut) u^{\mu-1} \phi(u) du \\ = 2 \int_0^\infty \left(\frac{x}{t}\right)^{\frac{1}{2}-\nu} \cdot \frac{\Gamma(1+\mu) f(x) t^{2\mu+2}}{(x^2+t^2)^{\mu+1}} dx \end{aligned}$$

or

$$\frac{(2t)^{-\mu-\frac{1}{2}}}{t^\nu} \cdot \pi^{\frac{1}{2}} \cdot \psi(\nu) = \int_0^\infty \frac{x^{3/2} f(x) \Gamma(1+\mu) x^{-\nu-1} dx}{(x^2+t^2)^{\mu+1}}$$

where

$$\psi(\nu) = \int_0^\infty (ut)^{\frac{1}{2}} J_{\mu+\nu}(ut) u^{\mu-1} \phi(u) du$$

Inverting this integral with the help of Inversion Theorem for Mellin Transforms we get under relevant conditions,

$$\Gamma(\mu+1) f(x) = \pi^{-\frac{1}{2}} \cdot x^{-\frac{\mu}{2}} \cdot (2t)^{-\mu-\frac{1}{2}} \cdot (t^2+x^2)^{\mu+1} \cdot \frac{1}{2i} \int_{k'-i\infty}^{k'+i\infty} \left(\frac{x}{t}\right)^\nu \psi(\nu) d\nu.$$

which is known [7, p. 739 (6.1)].

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## Applications of Jacobi Polynomials to Non-linear Oscillations

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### Abstract

The linear Jacobi polynomial approximation is applied to solve a non-linear differential equation

$$\ddot{x} + ax + bx^s = 0,$$

where  $s$  is a positive integer, generalizing the previously known results. The results are compared with various particular cases found in the literature. It is found that the approximation is good and compares satisfactorily with the exact values for  $s = 3$  and it can be improved by a suitable choice of the parameters of the polynomial.

### 1. Introduction

Denman<sup>1</sup> has shown that an amplitude-dependent approximation to the frequency of the simple pendulum may be obtained by making a linear Tchebycheff polynomial approximation to  $\sin\theta$  in the interval  $(-A, A)$ , where  $A$  is the amplitude of the motion. Denman and Howard<sup>2</sup> and Denman and Liu<sup>3</sup> have applied the ultraspherical polynomials, which include the Legendre polynomials, the Tchebycheff polynomials of the first and second kind, and the Tayler series as special cases, to the above problem. Garde<sup>4</sup> has applied Gegenbauer polynomials to some forced oscillation problems. In a recent paper<sup>5</sup> Jacobi polynomials are used to obtain a linear amplitude dependent approximation for the purpose of linearization of the non-linear differential equation of the type

$$\ddot{x} + ax + bx^s = 0$$

In the present paper the authors have applied the Jacobi polynomials to obtain the linear amplitude dependent approximate solution of the non-linear differential equation of a more general type

$$\ddot{x} + ax + bx^s = 0, \quad (1.1)$$

where  $s$  is a positive integer, subject to the initial conditions  $x = A$ ,  $\dot{x} = 0$  and  $t = 0$ ,  $A$  being the amplitude of motion. The results obtained are compared with those of special values of  $s$  available in the literature.

## 2. Jacobi polynomials and linear approximation.

Jacobi polynomials are the sets of polynomials orthogonal in the interval  $(-1, 1)$  with respect to the weight function  $(1-x)^\alpha (1+x)^\beta$ , where  $\operatorname{Re} \alpha > -1$  and  $\operatorname{Re} \beta > -1$ .

These polynomials may be defined by [6, p. 254]

$$P_n^{(\alpha, \beta)}(x) = \frac{(1+\alpha)_n}{n!} {}_2F_1 \left( -n, 1+\alpha+\beta+n; 1+\alpha; \frac{1-x}{2} \right), \quad (2.1)$$

where  ${}_2F_1$  is the Gauss's hypergeometric function.

Gegenbauer, Ultraspherical, Legendre, and Tchebycheff polynomials are particular cases of the polynomials (2.1).

In the interval  $(-A, A)$ , Jacobi polynomials are defined as the sets of polynomials orthogonal in this interval with respect to the weight function  $(1-x/A)^\alpha (1+x/A)^\beta$ . This gives rise to  $P_n^{(\alpha, \beta)}(x/A)$ .

For a function  $f(x)$ , which can be expanded in terms of Jacobi polynomials in the interval  $(-A, A)$ , we obtain

$$f(x) = \sum_{n=0}^{\infty} a_n^{(\alpha, \beta)} P_n^{(\alpha, \beta)}(x/A), \quad (2.2)$$

where the coefficients  $a_n^{(\alpha, \beta)}$  are given by

$$a_n^{(\alpha, \beta)} = \frac{\int_{-1}^1 f(Ax) P_n^{(\alpha, \beta)}(x) (1-x)^\alpha (1+x)^\beta dx}{\int_{-1}^1 [P_n^{(\alpha, \beta)}(x)]^2 (1-x)^\alpha (1+x)^\beta dx}. \quad (2.3)$$

If the series (2.2) is truncated after the second term, we obtain a linear approximation

$$f_*(x) = a_0^{(\alpha, \beta)} P_0^{(\alpha, \beta)}(x/A) + a_1^{(\alpha, \beta)} P_1^{(\alpha, \beta)}(x/A),$$

where star indicates approximation. The approximate period corresponding to linear approximation is given by

$$T_* = \frac{4\pi}{2+\alpha+\beta} \left( \frac{AB}{C} \right)^{\frac{1}{2}},$$

where

$$B = \frac{2^{1+\alpha+\beta} \Gamma(2+\alpha) \Gamma(2+\beta)}{(3+\alpha+\beta) \Gamma(2+\alpha+\beta)}$$

and

$$C = \int_{-1}^1 x f(Ax) (1-x)^\alpha (1+x)^\beta dx.$$

## 3. Application of Jacobi polynomials to non-linear differential equation.

In this section, we propose to solve the non-linear differential equation

$$\ddot{x} + f(x) = 0, \quad (3.1)$$

where

$$f(x) = ax + bx^s,$$

$s$  being a positive integer, by taking the linear Jacobi polynomials approximation of  $f(x)$ .

By § 2, the linear Jacobi polynomial approximation of  $f(x)$  gives

$$\begin{aligned} f_*(x) &= (ax + bx^s)_* = a_0^{(\alpha, \beta)} P_0^{(\alpha, \beta)}(x/A) + a_1^{(\alpha, \beta)} P_1^{(\alpha, \beta)}(x/A) \\ &= a_0^{(\alpha, \beta)} + a_1^{(\alpha, \beta)} \left[ \frac{\alpha - \beta}{2} + \frac{2 + \alpha + \beta}{2} \frac{x}{A} \right], \end{aligned} \quad (3.2)$$

because

$$P_0^{(\alpha, \beta)}(x) = 1, \quad P_1^{(\alpha, \beta)}(x) = \frac{\alpha - \beta}{2} + \frac{2 + \alpha + \beta}{2} x,$$

and where

$$a_0^{(\alpha, \beta)} = \frac{\int_{-1}^1 (aAx + bA^s x^s) P_0^{(\alpha, \beta)}(x) (1-x)^\alpha (1+x)^\beta dx}{\int_{-1}^1 [P_0^{(\alpha, \beta)}(x)]^2 (1-x)^\alpha (1+x)^\beta dx}, \quad (3.3)$$

and

$$a_1^{(\alpha, \beta)} = \frac{\int_{-1}^1 (aAx + bA^s x^s) P_1^{(\alpha, \beta)}(x) (1-x)^\alpha (1+x)^\beta dx}{c}, \quad (3.4)$$

where

$$c = 2^{1+\alpha+\beta} \Gamma(2+\alpha) \Gamma(2+\beta) / (3+\alpha+\beta) \Gamma(2+\alpha+\beta)$$

Using the integrals [7, p. 466], [6, p. 261], [8, p. 160],

$$\begin{aligned} & \int_{-1}^1 x^s (1-x)^\alpha (1+x)^\beta P_n^{(\alpha, \beta)}(x) dx \\ &= \frac{2^{n+\alpha+\beta+1} s! \Gamma(1+\alpha+n) \Gamma(1+\beta+s)}{n! (s-n)! \Gamma(n+s+\alpha+\beta+2)} \\ & \times {}_2F_1 \left( \begin{matrix} n-s, \alpha+n+1 \\ -\beta-s \end{matrix}, -1 \right), \text{ for } s > n \end{aligned} \quad (3.5)$$

$$= 0, \text{ for } s < n, \quad (3.6)$$

$$= \frac{2^{1+\alpha+\beta+n} \Gamma(1+\alpha+n) \Gamma(1+\beta+n)}{\Gamma(2+\alpha+\beta+2n)}, \text{ for } s=n. \quad (3.7)$$

We see that

$$\begin{aligned} a_0^{(\alpha, \beta)} &= \frac{(\beta - \alpha) aA}{\alpha + \beta + 2} + \frac{bA^s \Gamma(\alpha + \beta + 2) \Gamma(\beta + s + 1)}{\Gamma(s + \alpha + \beta + 2) \Gamma(1 + \beta)} \\ & \times {}_2F_1(-s, \alpha + 1; -\beta - s; -1), \end{aligned} \quad (3.8)$$

and

$$\begin{aligned} a_1^{(\alpha, \beta)} &= \frac{2aA}{2+\alpha+\beta} + \frac{2bsA^s (3+\alpha+\beta) \Gamma(2+\alpha+\beta) \Gamma(1+\beta+s)}{\Gamma(\alpha+\beta+s+3) \Gamma(2+\beta)} \\ & \times {}_2F_1 \left( \begin{matrix} 1-s, \alpha+2 \\ -\beta-s \end{matrix}, -1 \right). \end{aligned} \quad (3.9)$$

Replacing  $f(x)$  by its approximation  $f_*(x)$ , (3.1) transforms into

$$\ddot{x} + a_0^{(\alpha, \beta)} + a_1^{(\alpha, \beta)} \left\{ \frac{\alpha - \beta}{2} + \frac{2 + \alpha + \beta}{2} \frac{x}{A} \right\} = 0$$

or

$$\ddot{x} + \lambda_s^2 x = -\frac{(\alpha - \beta)A}{\alpha + \beta + 2} \left( \lambda_s^2 - \lambda_{1s}^2 \right), \quad (3.10)$$

where

$$\lambda_s^2 = a + b s A^{s-1} \frac{(3 + \alpha + \beta) \Gamma(\alpha + \beta + 3) \Gamma(1 + \beta + s)}{\Gamma(\alpha + \beta + s + 3) \Gamma(2 + \beta)} \times {}_2F_1 \left( \begin{matrix} 1 - s, \alpha + 2 \\ -\beta - s \end{matrix}; -1 \right), \quad (3.11)$$

$$\lambda_{1s}^2 = a - \frac{b A^{s-1} (\alpha + \beta + 2) \Gamma(\alpha + \beta + 2) \Gamma(1 + \beta + s)}{\Gamma(1 + \beta) \Gamma(s + \alpha + \beta + 2) (\alpha - \beta)} \times {}_2F_1 \left( \begin{matrix} -s, \alpha + 1 \\ -\beta - s \end{matrix}; -1 \right). \quad (3.12)$$

The approximate solution, subject to the initial conditions  $x = A$ ,  $\dot{x} = 0$  at  $t = 0$ , is given by

$$x_* = \left[ A + \frac{\alpha - \beta}{2 + \alpha + \beta} \left( 1 - \frac{2}{\lambda_{1s}^2} / \lambda_s^2 \right) A \right] \cos \lambda_s t - \frac{\alpha - \beta}{2 + \alpha + \beta} \left( 1 - \lambda_{1s}^2 / \lambda_s^2 \right). \quad (3.13)$$

The approximate period of oscillation is given by  $T_* = \frac{2\pi}{\lambda_s}$ .

#### 4. Discussion and comparison with the known results

##### Particular cases

(i)  $a = 0$ . For  $a = 0$ , we find that the approximate solution of the non-linear differential equation

$$\ddot{x} + b x^s = 0, \quad (4.1)$$

is given by

$$x_* = \left[ A + \frac{\alpha - \beta}{2 + \alpha + \beta} \left( 1 - \lambda_{1s}^{**2} / \lambda_s^{**2} \right) A \right] \cos \lambda_s^{**} t - \frac{\alpha - \beta}{2 + \alpha + \beta} \left( 1 - \lambda_{1s}^{**2} / \lambda_s^{**2} \right), \quad (4.2)$$

where

$$\lambda_s^{**2} = b s A^{s-1} \frac{\Gamma(\alpha + \beta + 4) \Gamma(1 + \beta + s)}{\Gamma(\alpha + \beta + s + 3) \Gamma(2 + \beta)} {}_2F_1 \left( \begin{matrix} 1 - s, \alpha + 2 \\ -\beta - s \end{matrix}; -1 \right) \quad (4.3)$$

$$\lambda_{1s}^{**2} = \frac{b A^{s-1} \Gamma(\alpha + \beta + 3) \Gamma(1 + \beta + s)}{\Gamma(1 + \beta) \Gamma(s + \alpha + \beta + 2) (\beta - \alpha)} {}_2F_1 \left( \begin{matrix} -s, \alpha + 1 \\ -\beta - s \end{matrix}; -1 \right). \quad (4.4)$$



The approximate period is given by

$$T_* = \frac{2\pi}{\lambda_s^{**}} \quad (4.5)$$

(ii) **b = 0 or s = 1.** For  $b = 0$  or  $s = 1$ , we obtain well known equation for S. H. M.

(iii) **s = 3.** For  $s = 3$ ,

$$\lambda_3^2 = a + 3bA^2 \frac{\{(\alpha - \beta)^2 + \alpha + \beta + 4\}}{(4 + \alpha + \beta)(5 + \alpha + \beta)} \quad (4.6)$$

and

$$\lambda_{13}^2 = a + \frac{bA^2 \{(\alpha - \beta)^2 + 3(\alpha + \beta) + 8\}}{(\alpha + \beta + 3)(\alpha + \beta + 4)} \quad (4.7)$$

and the approximate solution of

$$\ddot{x} + ax + bx^3 = 0, \quad (4.8)$$

is

$$x_* = \left[ A + \frac{\alpha - \beta}{2 + \alpha + \beta} \left( 1 - \frac{\lambda_{13}^2}{\lambda_8^2} \right) A \right] \cos \lambda_8 t - \frac{(\alpha - \beta)A}{2 + \alpha + \beta} \left( 1 - \frac{\lambda_{13}^2}{\lambda_8^2} \right), \quad (4.9)$$

and

$$T_* = 2\pi/\lambda_3, \quad (4.10)$$

which is the same as given by Garde [5, p. 112].

The exact solution of (4.8) is given as [9, p. 40]

$$x = AC_n \{ t(a + bA^2)^{\frac{1}{2}} \}, \quad (4.11)$$

where  $C_n(x)$  is the Jacobian elliptic function of modulus  $k$  given by

$$k^2 = \frac{bA^2}{2(a + bA^2)}.$$

The period of oscillation is

$$T = \frac{4F(k, \frac{1}{2}\pi)}{(a + bA^2)^{\frac{1}{2}}}. \quad (4.12)$$

where  $F(k, \frac{1}{2}\pi)$  is the complete elliptic integral of the first kind of modulus  $k$ .

In order to study how good our approximations compare with these exact solutions we consider the following two cases.

(a) As the approximation depends on the values of  $\alpha, \beta$  for a fixed values of  $a$  and  $b$ , we, therefore, consider the following values of  $\alpha, \beta$  for  $a = 10, b = 100, A = 1$ :

$$\alpha = -0.4, \beta = -0.5$$

$$\alpha = -0.5, \beta = -0.5$$

$$\alpha = +0.5, \beta = +0.5$$

The results are given in table I.

TABLE 1  
*Comparison of exact and approximate solutions for  $a = 10$ ,  $b = 100$ ,  $A = 1$*

$t$	Exact solution	Approximate solution for $\alpha = -0.4$ , $\beta = -0.5$	Approximate solution for $\alpha = -0.5$ , $\beta = +0.5$	Approximate solution for $\alpha = \beta = +0.5$
·01	·99501	·99591	·99595	·99680
·05	·87509	·88708	·89605	·92866
·06	·81914	·85398	·85252	·89605
·09	·63976	·68539	·67488	·77757
·12	·41877	·46346	·44645	·61375
·15	·20790	·20745	·18964	·41759
·16	·13058	·11945	·10063	·34365

(b) To study how our approximations compare for different values of  $a$  and  $b$  for the same values of  $\alpha$  and  $\beta$ , we have considered the following values of  $a$  and  $b$  for  $\alpha = -0.4$  and  $\beta = -0.5$

(i)  $a = b = 10$

(ii)  $a = 100$ ,  $b = 10$

The results are given in Tables 2 and 3.

TABLE 2  
*Comparison of exact and approximate solution for  $a = b = 10$*

$t$	Exact solution	Approximate solution
·01	·99920	·99920
·05	·97597	·97811
·06	·96398	·96913
·09	·92206	·93260
·12	·86077	·87806
·15	·79090	·81462
·20	·64868	·67618
·25	·47607	·50819
·30	·29635	·31805
·35	·10879	·11412

TABLE 3  
*Comparison of exact and approximate solution for  $a = 100$ ,  $b = 10$*

$t$	Exact solution	Approximate solution
·01	·99500	·99503
·05	·86782	·86861
·06	·80916	·81499
·09	·60375	·60024
·12	·31731	·32885
·14	·11770	·12577

(iv)  $s = 2$ . For  $s = 2$ , the exact solutions are not available. This case has, however, been considered in [9, p. 51(16)] for an approximate solution. The solution is as follows:

$$x = - \left[ (bA^2/2w^2) + (b^2A^3/3w^4) \right] + \left[ A + bA^2/3w^2 + \frac{29b^2A^3}{576} \right] \cos wt + \left[ \frac{bA^2}{6w^2} + \frac{b^2A^3}{9w^4} \right] \cos 2wt + \frac{b^2A^3}{48w^4} \cos 3wt, \quad (4.13)$$

with

$$w \simeq a^{\frac{1}{2}} \left\{ 1 - \frac{5b^2A^2}{12a^2} \right\} \text{ and period} = \frac{2\pi}{w}. \quad (4.14)$$

From our study (3.13) it follows that

$$x_* = A \left[ 1 + \frac{\left\{ 1 - \frac{(\alpha - \beta)^2}{2 + \alpha + \beta} \right\}}{a(\alpha + \beta + 4) + 2bA(\beta - \alpha)} \right] \cos \lambda_2 t - A \left[ 1 - \frac{(\alpha - \beta)^2}{2 + \alpha + \beta} \right] / a(\alpha + \beta + 4) + 2bA(\beta - \alpha), \quad (4.15)$$

where

$$\lambda_2^2 = a + \frac{2bA(\beta - \alpha)}{\alpha + \beta + 4},$$

and the approximate period

$$T_* = \frac{2\pi}{\lambda_2}. \quad (4.16)$$

In order to compare (5.1) and (5.3) numerically we take the following special values of the parameters as given in [9, p. 248, ex. 18], namely

$a = 400$ ,  $b = 100$ ,  $A = 1$  and one of our values of  $\alpha$  and  $\beta$ , i.e.  $\alpha = -0.4$ ,  $\beta = -0.5$ .

The solution (5.1) becomes

$$x = -0.155 + 1.102 \cos wt + .052 \cos 2wt + .001 \cos 3wt \quad (4.17)$$

with  $w = 19.45$  and the period

$$= \frac{2\pi}{w} = \frac{2\pi}{19.45} = .322, \quad (4.18)$$

and (5.3) becomes

$$x_* = \cos(19.84)t \quad (4.19)$$

and

$$T_* = \frac{2\pi}{19.84} = .321.$$

The solutions (4.17) and (4.19) are compared below in Table 4.

TABLE 4

$t$	$x_*$ (eq. 4.19)	$x$ (eq. 4.17)
·01	·98007	·97499
·02	·92491	·90198
·03	·83370	·78580
·04	·70171	·63010
·05	·54869	·48464
·06	·37166	·24286

### Conclusions

In the present paper the linear Jacobi polynomial approximation of the non-linear restoring force has been applied in the linearisation of the non-linear differential equation involving general power of  $x$ . The results thus obtained are found to agree fairly with the known exact results in cases  $s = 1$  and  $s = 3$ . The approximation can be improved by suitable choice of the parameters  $\alpha$  and  $\beta$ .

From table 1 it can be seen that the variation of  $\beta$  effects the solution slowly but on the other hand the variation in  $\alpha$  is quite effective. It appears that low values of  $\alpha$  would be necessary for approaching the exact solution. From tables 2 and 3 it is obvious that coefficient  $a$  is more effective than the coefficient  $b$  which means that the effect of the first term of the equation will always dominate in the result. It means that the perturbations will always remain small.

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# Resultant of Kernels in Two Types of Transformations

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## Abstract

In this paper two types of transformations, given by Charles Fox in T-chains, are studied with the help of resultant of kernel functions.

## 1. Introduction

Charles Fox [6, 7] has established T-chains in two types of Transformations. One is given by

$$(1.1) \quad g_{m+1}(x) = \int_0^\infty g_m(u) k_m(ux) du \quad [m = 1, \dots, n]$$

and other is of the type

$$(1.2) \quad g_{i+1}(x) = \int_0^\infty r_i\left(\frac{x}{u}\right) g_i(u) \frac{du}{u} \quad (i = 1, \dots, p]$$

$$(1.3) \quad \int_0^\infty l_j\left(\frac{x}{u}\right) g_{j+1}(u) \frac{du}{u} = g_j(x) \quad [j = 1, \dots, q]$$

where  $p+q = n$ ,  $r_i(x)$  and  $l_j(x)$  are the kernels. If  $g_{n+1}(x) = g_1(x)$  then (1.1), (1.2) and (1.3) form a T-chain. The T-chain may consist of only one type of kernels.

The object of this paper is to study these transformations given by (1.1) and (1.2) and T-chains given by Fox, with the help of the resultant of two kernel functions.

In section 2, we have discussed the resultant of two kernels for each of the two transformations. In § 3 the properties of the resultant kernel functions are compared with the properties of its components for each type of transformation, leading to certain conjectures. With the help of these conjectures, the resultants of symmetrical and unsymmetrical Fourier kernels are obtained in § 4. In § 5 the resultant of sets of kernels is obtained while in § 6, the resultant of T-chains is discussed. In § 7 some of the conjectures obtained in § 3 are proved and are illustrated.

## 2. Definition of resultant of two kernels and two type of Transformations :

(a) Resultant of two kernels. In equations given by

$$(2.1) \quad g_2(x) = \int_0^\infty g_1(u) k_1(xu) du$$

$$(2.2) \quad g_3(y) = \int_0^{\infty} g_2(x) k_2(xy) dx$$

the kernels  $k_1(xu)$  and  $k_2(xy)$  are the product kernels. [3, page 5]. We shall refer this type of transformation as transformation I.

We define the resultant of the two kernels  $k_1(xu)$  and  $k_2(yu)$  as a function,  $R(u, y)$ , which transforms  $g_1(u)$  to  $g_3(y)$  and  $k_1(xu)$ ,  $k_2(xy)$  as the component kernels. Thus the resultant given by

$$(2.3) \quad R(u, y) = \int_0^{\infty} k_1(xu) k_2(xy) dx$$

is such that

$$(2.4) \quad g_3(y) = \int_0^{\infty} g_1(u) R(u, y) du$$

Also 
$$R(u, y) = \frac{1}{u} r_1\left(\frac{y}{u}\right)$$

where

$$(2.5) \quad r_1(\eta) = \int_0^{\infty} k_1(\xi) k_2(\xi\eta) d\xi$$

If  $g_1(u)$  is transformed by  $k_2(xu)$  first and then by  $k_1(xy)$ , the resultant

$$(2.6) \quad \begin{aligned} R'(u, y) &= \int_0^{\infty} k_2(xu) k_1(yx) dx \\ &= \frac{1}{y} r_1\left(\frac{u}{y}\right) \\ &= R(y, u) \end{aligned}$$

showing that the resultant remains unchanged if the function  $R(u, y)$  is symmetrical with respect to  $u$  and  $y$ .

Conditions for the existence of the resultant kernel can be obtained as in [6, page 230].

(A) If

(i)  $\int_0^{\infty} |k_1(x)|^2 dx$  exists

(ii)  $K_2(s)$  is bounded on the line  $s = \frac{1}{2} + it$  for all real values of  $t$

(iii) differentiation with respect to  $x$  through integral sign is permissible for the integral

$$\int_0^{\infty} k_1(u) \bar{k}_2(xu) \frac{du}{u}$$

then

(a)  $r_1(x) \in L(0, \infty)$ , and

$$(2.7) \quad r_1(x) = \int_0^{\infty} k_1(u) k_2(xu) du$$

where

$$(2.8) \quad \bar{k}(x) = \int_0^{\infty} k(u) du$$

and

$$(2.9) \quad K(s) = \int_0^\infty k(u) u^{s-1} du$$

(B) If (i) and (ii) of (A) hold good while (iii) is not true then

$$(2.10) \quad \bar{r}_1(x) = \int_0^\infty k_1(u) \bar{k}_2(xu) \frac{du}{u}$$

2(b) Now consider the equations

$$(2.11) \quad g_2(x) = \int_0^\infty g_1(u) k_1\left(\frac{x}{u}\right) \frac{du}{u}$$

$$(2.12) \quad g_3(y) = \int_0^\infty g_2(x) k_2\left(\frac{y}{x}\right) \frac{dx}{x}$$

where the kernels  $\frac{1}{u} k_1(x/u)$  and  $\frac{1}{x} k_2(y/x)$  are in the quotient form. We shall refer this type as Transformation II. If  $g_1(x) = g_3(x)$ , then  $k_1(x)$  and  $k_2(x)$  will form a pair of unsymmetrical Fourier kernels in Transformation II.

In Transformation II, the resultant of two kernels  $\frac{1}{u} k_1(x/u)$  and  $\frac{1}{x} k_2(y/x)$ , by definition, is a function  $R(u, y)$  which transforms  $g_1(u)$  to  $g_3(y)$ .

Thus the resultant given by

$$(2.13) \quad R(u, y) = \int_0^\infty \frac{1}{u} k_1\left(\frac{x}{u}\right) \frac{1}{x} k_2\left(\frac{y}{x}\right) dx$$

is such that

$$(2.14) \quad g_3(y) = \int_0^\infty g_1(u) R(u, y) du$$

Also

$$R(u, y) = \frac{1}{u} r_2(y/u)$$

where

$$(2.15) \quad r_2(\eta) = \int_0^\infty k_1(\xi) k_2(\eta/\xi) \frac{d\xi}{\xi}$$

If  $g_1(u)$  is transformed by  $\frac{1}{u} k_2(x/u)$  first and then by  $\frac{1}{x} k_1(y/x)$ , the resultant

$$(2.16) \quad R'(u, y) = \int_0^\infty \frac{1}{u} k_2\left(\frac{x}{u}\right) \frac{1}{x} k_1\left(\frac{y}{x}\right) dx$$

Replacing  $y/x$  by  $z$  and using (2.15) we get

$$(2.17) \quad R'(u, y) = \frac{1}{u} r_2\left(\frac{y}{u}\right) = R(u, y)$$

This shows that the resultant remains unchanged even if the kernels  $\frac{1}{u} k_1(x/u)$

and  $\frac{1}{x} k_2(y/x)$  are interchanged. Conditions for the existence of the resultant kernel in this type of transformation can be obtained as in [11, Theorem 28, page 46, Theorem 44, page 60] and in [7, page 680]

(C) We have, if

- (i)  $y^{-\frac{1}{2}} k_1(y) \in L(0, \infty)$
- (ii)  $y^{-\frac{1}{2}} k_2(y) \in L(0, \infty)$
- (iii)  $y^{-\frac{1}{2}} k_2(y)$  is of bounded variation in the neighbourhood of  $y = x$

then

$$(a) \quad y^{-\frac{1}{2}} r_2(y) \in L(0, \infty)$$

where  $r_2(x)$  is as in (2.15).

(D) Again, [7, page 684] if

- (i)  $k_1(y) \in L^2(0, \infty)$
- (ii)  $K_2(s)$  is bounded on the line  $s = \frac{1}{2} + it$  for all real values of  $t$
- (iii) differentiation with respect to  $x$  through integral sign is permissible for the integral

$$\int_0^\infty k_1(y) \bar{k}_2(x/y) dy$$

then

$$(a) \quad r_2(x) \in L^2(0, \infty)$$

where  $r_2(x)$  is as in (2.15)

(E) If (i) and (ii) of (D) hold good, while (iii) is not true, then

$$(2.18) \quad \bar{r}_2(x) = \int_0^\infty k_1(y) \bar{k}_2(x/y) dy$$

3. *Properties of two types of Transformations* : —In this section we now compare the properties of two types of transformations I and II as observed in section 2, with particular reference to the nature of the resultant kernel and its components.

- (i) The kernels  $k_1(xu)$  and  $k_2(xy)$  in Transformation I are product kernels while in Transformation II they are in the form  $\frac{1}{u} r_2(y/u)$ , a quotient form
- (ii) The forms of resultant kernels in Transformation I and II are similar. viz.  $\frac{1}{u} r_1(y/u)$  and  $\frac{1}{u} r_2(y/u)$
- (iii) The resultant kernel  $R(u, y)$  in (2.3) and  $R'(u, y)$  in (2.6) in Transformation I are generally different, except for the symmetry, with respect to  $u$  and  $y$ . In Transformation II, the two resultants are same.
- (iv) From conditions (A) and (B) of § 2 we observe that conditions on  $k_1(x)$  and  $k_2(x)$  for existence of  $\frac{1}{u} r_1(y/u)$  and those exhibited by  $\frac{1}{u} r_2(y/u)$



are different in Transformation I, while in Transformation II, it can be seen from conditions (C) and (D) of § 2, that the conditions imposed on  $k_1(x)$  and  $k_2(x)$  give rise to similar conditions on  $r_2(x)$ , which in turn give rise to similar behaviour of  $\frac{1}{u} k_1(x/u)$ ,  $\frac{1}{x} k_2(y/x)$  and  $\frac{1}{u} r_2(y/u)$ . If  $k_1(x)$  and  $k_2(x)$  denote the kernels of Transformation II, then  $r_2(x)$  will denote the resultant kernel.

These observations suggest us the following conjectures.

- (a) Nature of the resultant in Transformation I will generally be different from the nature of its components.
- (b) In Transformations II, we expect that the properties of  $\frac{1}{u} k_1(x/u)$  and  $\frac{1}{x} k_2(y/x)$  should be present in  $\frac{1}{u} r_2(y/u)$  and the properties of  $k_1(x)$  and  $k_2(x)$  in  $r_2(x)$ .
- (c) A transformation affected by two kernels  $k_1(xu)$  and  $k_2(xy)$  in Transformation I is equivalent to a single transformation affected by the resultant kernel  $\frac{1}{u} r_1(y/u)$  in Transformation II. Thus (2.1) and (2.2) can be replaced by a single equation.

$$(3.1) \quad g_3(y) = \int_0^\infty g_1(u) \frac{1}{u} r_1(y/u) du$$

of Transformation II, without changing the order of (2.1) and (2.2)

- (d) All T-chains are equivalent to one another.

4. *Resultant of Symmetrical and Unsymmetrical Fourier Kernels.* In Transformation I, if  $\frac{1}{u} r_1(y/u)$  is the resultant of  $k_1(xu)$  and  $k_2(xy)$  then under the conditions (A) and

(B) of § 2, the Mellin transform,  $R_1(s)$  of  $r_1(x)$  as in (2.5), can be written as

$$(4.1) \quad R_1(s) = K_1(1-s) K_2(s).$$

By taking Inverse Mellin transform of  $R_1(s)$ ,  $r_1(x)$  can be obtained.

If  $k_1(x)$  and  $k_2(x)$  form a pair of unsymmetrical Fourier kernels then [8, page 116]

$$(4.2) \quad K_1(1-s) K_2(s) = 1$$

giving  $R_1(s) = 1$ ,  $r_1(x)$  will exist as in (2.10) [8, page 123] and is given by

$$(4.3) \quad \left. \begin{aligned} r(x) &= 0 & (0 < x < 1) \\ &= 1 & (x > 1) \end{aligned} \right\}$$

In the case of the symmetrical Fourier kernels,  $k_1(x) = k_2(x)$  and  $r_1(x)$  is same as in (4.3)

Because the resultant,  $r_1(x)$  does not exist as in (2.5), so these cannot be replaced by a single equation, as (2.1).

In Transformation II, under conditions (C) and (D) of § 2 we get

$$(4.4) \quad R_2(s) = K_1(s) K_2(s)$$

instead of (4.1)  $r_2(x)$  can be obtained by taking Inverse Mellin transform of  $R_2(s)$

If we denote by  $k_u(x)$ , a unit kernel which transforms  $g_1(x)$  into itself, we see then  $K_u(s)$ , its Mellin transform is unity. Hence the kernel  $k_u(x)$  is same as  $r_1(x)$  which exists only as in (4.3)

Also in the case of symmetrical or unsymmetrical Fourier kernels

$$(4.5) \quad R_2(s) = K_1(s) K_2(s) = 1$$

so  $r_2(x)$  is same as  $k_u(x)$  and exists only as in (2.10). Thus two equations containing Fourier Symmetrical or Unsymmetrical kernels cannot be replaced by a single equivalent relation of the same form as (2.11)

5. Resultant of a set of kernels : — In this section we shall consider the resultant of three types of sets of kernels.

(a) Resultant of  $n$  kernels of Transformation II : We consider a set of equations say  $n$  in number given by

$$(5.1) \quad g_{p+1}(x) = \int_0^\infty g_p(y) k_p(x/y) \frac{dy}{y} \quad [p = 1, \dots, n]$$

transforming a function  $g_1(y)$  into  $g_{n+1}(y)$  through  $n$  integrations. Under most favourable conditions on  $g_1(y)$  and  $k_p(y)$  as in (C) and (D) of § 2, we see that the first two equations can be replaced by a single equation

$$(5.2) \quad g_3(x) = \int_0^\infty g_1(y) Kr_1(x/y) \frac{dy}{y}$$

where  $Kr_1(y)$  is the resultant of  $k_1(y)$  and  $k_2(y)$  reducing  $n$  equations of (5.1) to  $(n-1)$ . Again equation (5.2) and no. 3 of (5.1) can be replaced by a single equation

$$(5.3) \quad g_4(x) = \int_0^\infty g_1(y) Kr_{r_2}(x/y) \frac{dy}{y}$$

where  $Kr_{r_2}(y)$  is the resultant of  $Kr_1(y)$  and  $k_3(y)$  reducing to  $(n-2)$  equations. Repeating this process the set of  $n$  equations of (5.1) can be replaced by a single equation of the same type, transforming  $g_1(y)$  to  $g_{n+1}(y)$  with the help of a single resultant kernel. This single resultant kernel can also be found with the help of Mellin transform relation, without going through  $n$  different operations of integrations. If  $r_2(x)$  is the resultant of this set of kernels,  $k_p(x)$  then [7, page 680]

$$(5.4) \quad R_2(s) = \prod_1^n K_p(s)$$

and  $r_2(x)$  will be obtained by taking Inverse Mellin transform of  $R_2(s)$

As an example consider a set of  $m$  kernels given by  $x^{\frac{1}{2}} J_{\nu_r}(x)$  [ $r = 1, 2, \dots, m$ ]; then we observe that each kernel satisfies conditions (C) of § 2 if  $\nu_r > -\frac{1}{2}$ . Using (5.4) we have

$$(5.5) \quad R_2(s) = 2^{m(s-\frac{1}{2})} \prod_{r=1}^m \frac{\Gamma\left(\frac{\nu_r}{2} + \frac{s}{2} + \frac{1}{4}\right)}{\Gamma\left(\frac{\nu_r}{2} - \frac{s}{2} + \frac{3}{4}\right)}$$

Taking Inverse Mellin transform of  $R_2(s)$  we get  $r_2(x)$ , the kernel  $\omega_{\nu_1, \nu_2, \dots, \nu_m}(x)$  defined by Bhatnagar [1]. In particular if  $\nu_1 = \mu$  and  $\nu_2 = \nu$  we get Watson's kernel, [12, page 308]  $\omega_{\mu, \nu}(x)$ .

(b) *Resultant of  $2n$  kernels of Transformation I*: Consider a set of  $2n$  equations of Transformation I transforming  $g_1(x)$  to  $g_{2n+1}(x)$ . Suppose that  $r_{(q+1)/2}(x)$ , the resultant of  $k_q(x)$  and  $k_{q+1}(x)$  [ $q = 1, 3, 5, \dots, (2n-1)$ ] exist as in (2.5). Each pair of equations containing kernels  $k_q(x)$  and  $k_{q+1}(x)$  can be replaced by a single equation, under conditions (A) of § 2 as shown in § 3 (C), thus reducing a set of  $2n$  equations of Transformation I to  $n$  equations of Transformation II. Further under conditions (D) of § 2 it can be replaced by a single equivalent relation as in § 5 (a).

The resultant can be obtained by using Mellin Transform relation

$$(5.6) \quad R(s) = \prod_{p=1}^n K_{2p-1}(s) K_{2p}(1-s)$$

similar to (5.4).

As an example consider a set of 4 kernels, each  $e_1^{-x}$  of Transformation I, Under the conditions (A) of § 2,  $r_1(x)$  as in (2.5) exists and is given by  $1/(1+x)$ . Further  $x^{-\frac{1}{2}}/(1+x) \in L(0, \infty)$  and is of bounded variation in the neighbourhood of  $x = y$  ( $y \neq -1$ ). Thus  $r_2(x)$  as in (2.15) is given by

$$(5.7) \quad r_2(x) = \int_0^\infty \frac{1}{(1+y)} \frac{1}{y} \frac{1}{(1+x/y)} dy$$

Using [5, page 216], we have

$$(5.8) \quad r_2(x) = (x-1)^{-1} \log x \quad x \neq -1$$

Using Mellin transform relation (5.6), we have

$$(5.9) \quad R(s) = \Gamma(s) \Gamma(1-s) \Gamma(s) \Gamma(1-s) = \pi^2 \operatorname{cosec}^2 \pi s$$

Usnig [4, page 346],  $r_2(x)$  is same as in (5.8).

(c) *Resultant of  $(2n+1)$  kernels of Transformation I*:—We consider a set of  $(2n+1)$  equations of Transformation I, transforming  $g_1(x)$  to  $g_{2n+2}(x)$ . Of these the first  $2n$  equations can be replaced by a single equivalent equation of transformation II, under favourable conditions. Thus the set can be reduced to two equations, one of Transformation I and other of Transformation II, as

$$(5.10) \quad g_{2n+1}(x) = \int_0^\infty g_1(y) r\left(\frac{x}{y}\right) \frac{dy}{y}$$

$$(5.11) \quad g_{2n+2}(z) = \int_0^\infty g_{2n+1}(x) k_{2n+1}(xz) dx$$

where  $\frac{1}{y} r(x/y)$  is the resultant kernel of first  $2n$  kernels of the set. Now (5.10) and (5.11) can be further reduced to a single equation if the integral

$$(5.12) \quad \int_0^\infty r\left(\frac{x}{y}\right) k_{2n+1}(xz) dx$$

can be evaluated.

The resultant can also be obtained by using Mellin transform relation

$$(5.13) \quad R(s) = K_{2n+1}(s) \prod_{q=1}^n K_{2q-1}(s) K_{2q}(1-s)$$

## 6. Resultant of T-chains :

We shall obtain the resultant of three types of T-chains discussed by Fox.

A set of  $n$  equations in Transformation I and II forms a T-chain if  $g_i(x) = g_{n+i}(x)$ . This gives  $R(s) = 1$  in (5.4), (5.6) and (5.13).  $r(x)$  cannot be obtained as  $r_1(x)$  in (2.5) or  $r_2(x)$  in (2.15). It is identical with  $K_u(x)$ . Thus transformation of a T-chain is equivalent to that of a unit kernel of Transformation II.

Consider a T-chain of  $n$  equations of transformation II. The first  $(n-1)$  equations do not form a chain but a set, so under sufficient conditions it can be replaced by a single equation as shown in § 5 (a). This equation obtained and the  $n$ th equation left, form a pair of equations containing unsymmetrical Fourier kernels. These cannot be further replaced by a single equation as shown in § 4.

Consider a T-chain of  $2n$  equations of Transformation I. As shown in § 5(a) these can be replaced in general by  $n$  equations of Transformation II and ultimately by a pair of equations containing unsymmetrical Fourier kernels.

We now consider a T-chain of  $(2n+1)$  equations of Transformation I. The first  $2n$  kernels do not form a chain and can be replaced in general by a single equation of Transformation II. So in this case  $(2n+1)$  equations of Transformation I are equivalent to two equations, one of Transformation I and the other of Transformation II. This leads to a existence of a Mixed T-chain and a Mixed set.

Generally all the kernels of a set or of a T-chain will not obey favourable conditions. But if some of them, obey these conditions, then this can be used to reduce the length of a T-chain without affecting the total transformation. e.g. in a six term K-chain of Transformation I [6, page 231] given by

$$k_4(x) = k_1(x) = \sqrt{\frac{2}{\pi}} \sin x, \quad k_5(x) = k_2(x) = \sqrt{\frac{2}{\pi}} \cos x$$

$$k_6(x) = k_3(x) = x^{\frac{1}{2}} J_0(x)$$

the resultants of  $k_1(x)$ ,  $k_2(x)$  and also of  $k_4(x)$  and  $k_5(x)$  exist. Thus we have only four equations forming a Mixed T-chain with kernels  $\bar{k}_r(x)$ , ( $r = 1, 2, 3, 4$ ) given by  $\bar{k}_1(x) = \bar{k}_3(x) = \frac{2}{\pi(1-x^2)}$ , ( $x \neq -1$ ),  $\bar{k}_2(x) = \bar{k}_4(x) = x^{\frac{1}{2}} J_0(x)$  where  $\bar{k}_1(x)$  and  $\bar{k}_3(x)$  are of Transformation II and  $\bar{k}_2(x)$  and  $\bar{k}_4(x)$  are of Transformation I.

If we consider a mixed set of four kernels,  $\cos x$ ,  $\sin x$  of Transformation II and  $e^{-x}$ ,  $e^{-ix}$  of Transformation I, then these are equivalent to two kernels, [7, page 687]  $\frac{\pi}{2} J_0(x^{\frac{1}{2}})$  and  $1/(1+x)$  of Transformation II. Thus a mixed set of 4 equations is reduced to two equations of Transformation II.

#### 7. Properties of resultant kernels and its components :

In this section we show that some properties of the component kernels are observed in the resultant kernel.

##### (a) Theorem 1.

Under the conditions (A), (B), (C) and (D) of § 2 if

- (i)  $k_1(x)$  and  $k_2(x)$  are different Fourier kernels in Transformation I
- (ii)  $r_2(x)$  is the resultant of  $k_1(x)$  and  $k_2(x)$  in Transformation II as in (2.15)

then  $r_2(x)$  will also be a Fourier kernel in Transformation I.

*Proof:*

Under the conditions (A), (B), (C) and (D) of § 2, the Mellin transform relations corresponding to (i) and (ii) of the theorem will be

$$(7.1) \quad K_1(s) K_1(1-s) = 1, K_2(s) K_2(1-s) = 1 \quad [8, \text{page 116}]$$

$$(7.2) \quad R_2(s) = K_1(s) K_2(s)$$

Using (7.1) and (7.2) we have

$$(7.3) \quad R_2(s) R_2(1-s) = 1$$

showing that  $r_2(x)$  is a Fourier kernel.

This proves the theorem.

For example, the resultant kernels  $\omega_{\mu, \nu}(x)$  and  $\omega_{\nu_1, \nu_2 \dots \nu_m}(x)$  obtained in § 5(a) are known to be Fourier kernels.

##### (b) Theorem 2

Under the conditions (A), (B) and (D) of § 2 if

$$(7.4) \quad (i) \quad g_i(y) = \int_0^\infty f_i(x) \frac{1}{x} k_i(y/x) dx, \quad (i = 1, 2)$$

- (ii) resultants  $g(y)$  of  $g_1(y)$  and  $g_2(y)$ ,  $f(y)$  of  $f_1(y)$  and  $f_2(y)$  and  $k(y)$  of  $k_1(y)$  and  $k_2(y)$  of transformation II as in (2.15) exist

then  $g(y)$  will be  $k$ -transform of  $f(y)$

*Proof:*

Under the conditions (A), (B) and (D) of § 2 the Mellin transform of relations corresponding to equations of (i) and (ii) give

$$(7.5) \quad G_i(s) = K_i(s) F_i(s) \quad (i = 1, 2)$$

$$(7.6) \quad G(s) = G_1(s) G_2(s), F(s) = F_1(s) F_2(s) \text{ and}$$

$$K(s) = K_1(s) K_2(s)$$

Eliminating  $K_1(s), K_2(s), F_1(s), F_2(s), G_1(s)$  and  $G_2(s)$

we get

$$(7.7) \quad G(s) = F(s) K(s)$$

The equivalent relation to (7.7) is

$$(7.8) \quad g(x) = \int_0^\infty f(u) \frac{1}{u} k\left(\frac{x}{u}\right) du$$

This proves the theorem.

A similar theorem, stated below, can be proved on similar lines as in Theorem 2.

*Theorem 3.*

Under conditions (A), (B) and (D) of § 2 if

(i)  $g_1(y)$  is the  $k$ -transform of  $f_1(y)$ ,  $g_2(y)$  is the  $k_2$ -transform of  $f_2(y)$  in Transform I

(ii) resultants  $g(y)$  of  $g_1(y)$  and  $g_2(y)$ ,  $f(y)$  of  $f_1(y)$  and  $f_2(y)$  and  $k(y)$  of  $k_1(y)$  and  $k_2(y)$  of Transformation II as in (2.15) exist

then

$g(y)$  will be  $k$ -transform of  $f(y)$  in Transformation I.

In particular if  $f_1(y)$  and  $f_2(y)$  are self-reciprocal in  $k_1$  and  $k_2$ -transforms respectively then  $f(y)$  will be self-reciprocal in  $k$ -transform.

As an example, it can be seen that if  $f_1(y)$  is  $R_\mu$  [I1; page 245] and  $f_2(y)$  is  $R_\nu$ , then

$$(7.9) \quad f(x) = \int_0^\infty f_1(y) \frac{1}{y} f_2\left(\frac{x}{y}\right) dy$$

is self-reciprocal in  $\omega_{\mu, \nu}$ -transform [9, page 198].

(c) We now consider the inverse problem of finding one component kernel when the resultant kernel and one of its component kernels is known

Given the kernel  $k_1(x)$  and the resultant kernel  $r(x)$ , if

$$(7.10) \quad g_2(x) = \int_0^\infty g_1(y) \frac{1}{y} k_1\left(\frac{x}{y}\right) dy$$

$$(7.11) \quad g_3(x) = \int_0^\infty g_1(y) \frac{1}{y} r\left(\frac{x}{y}\right) dy$$

then we have to obtain the component kernel,  $k_2(y)$ , transforming  $g_2(y)$  into  $g_3(y)$  as in

$$(7.12) \quad g_3(x) = \int_0^\infty g_2(y) \frac{1}{y} k_2\left(\frac{x}{y}\right) dy$$

Assuming the conditions on  $g_1(x)$ ,  $g_2(x)$ ,  $g_3(x)$ ,  $k_1(x)$ ,  $k_2(x)$ , and  $r(x)$  allow us to take Mellin transforms of  $g_1(x)$ ,  $\dots$ ,  $r(x)$ , we obtain (4.4) giving

$$(7.13) \quad K_2(s) = R(s)/K_1(s)$$

Taking Inverse Mellin transform of  $K_2(s)$ , we obtain  $k_2(x)$  as

$$(7.14) \quad k_2(x) = \frac{1}{2\pi i} \lim_{T \rightarrow \infty} \int_{\frac{1}{2} - iT}^{\frac{1}{2} + iT} \frac{R(s)}{K_1(s)} x^{-s} ds.$$

Further if  $k_1(x)$  is one of the kernels of a pair of unsymmetrical or symmetrical Fourier kernels of Transformation I, we can write [8, page 117, 118]

$$(7.15) \quad K_1(s) H_1(1-s) = 1$$

where  $H_1(s)$  is the Mellin transform of the other kernel of the pair of kernels in equations given by

$$(7.16) \quad g(y) = \int_0^\infty f(x) k_1(xy) dx$$

$$(7.17) \quad f(y) = \int_0^\infty g(x) h_1(xy) dx$$

Then (7.14) becomes

$$(7.18) \quad k_2(x) = \frac{1}{2\pi i} \lim_{T \rightarrow \infty} \int_{\frac{1}{2} - iT}^{\frac{1}{2} + iT} R(s) H_1(1-s) x^{-s} ds.$$

Using Parseval relation for Mellin transform [11, page 54]

we get

$$(7.19) \quad k_2(x) = \int_0^\infty r(xy) h_1(y) dy$$

If we select  $r(x) = \omega_{\mu, \nu}(x)$ ,  $h_1(x) = k_1(x) = x^{\frac{1}{2}} J_\mu(x)$  then by (7.19)

$$(7.20) \quad k_2(x) = \int_0^\infty \omega_{\mu, \nu}(xy) y^{\frac{1}{2}} J_\mu(y) dy = x^{\frac{1}{2}} J_\nu(x)$$

This suggests that if  $f_1(y)$  has a property  $A$  with respect to kernel  $k_1(y)$  and  $f(y)$ , the resultant of  $f_1(y)$  and  $f_2(y)$ , has the same property  $A$ , with respect to the resultant kernel,  $r(y)$ , then  $f_2(y)$  will have property  $(A)$  with respect to  $k_2(y)$ .

We here state a result obtained by Bhatnagar [2, page 51] in its support.

If  $f(x)$  is  $R_{\mu, \nu}$  and belongs to  $A(\omega, a)$  [11, page 252] and  $\rho(x)$  is  $R_\mu$ , then

$$(7.21) \quad g(x) = \int_0^\infty f(zx) \rho(z) dz$$

is  $R_\nu$

Here  $g(x)$  is to be interpreted as component of  $f(x)$

In an earlier paper [10] the author has obtained, symmetrical Transforming kernels, changing one self-reciprocal function in one transform to another self-reciprocal function in another transform. The transforming kernels are the resultants of two self reciprocal functions in different transform and the conversion of one self reciprocal function into another amounts to finding the other component of the resultant when one self-reciprocal function is known.

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## Influence of different organic matter and phosphate on the reclamation of alkali soil in the presence and absence of light

By

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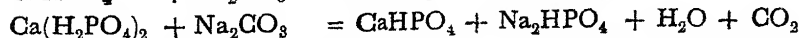
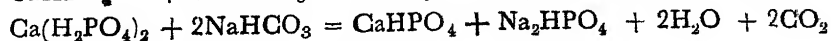
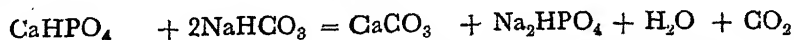
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### Abstract

When different organic substances like starch, water hyacinth, and cactus are mixed with alkali soil, they are slowly oxidised and in this process, a marked increase in total nitrogen, exchangeable calcium and available phosphate and a marked decrease in pH and electrical conductivity takes place resulting in permanent reclamation of alkali-soil. These changes become much more pronounced when organic substances are reinforced with basic slags like German basic slag and Tata basic slag and rock phosphate like Bihar rock phosphate. Moreover, the efficiency of the process of reclamation by the mixture of organic matter and calcium phosphate is greatly enhanced in the presence of light.

Reclamation of alkali soils presents a problem of immense practical importance for bringing back millions of acres of such unproductive lands once again under the plough. Dhar<sup>1</sup> has developed a new concept for the reclamation of alkali soil by a mixture of organic matter and calcium phosphates. According to him the basic principles involved in the method of reclaiming alkali land by a mixture of organic matter and calcium phosphates are (i) the creation of humus in soil and humus has excellent buffer qualities and (ii) the formation of di- and small amounts of monocalcium phosphates by the action of carbonic acid and organic acids produced in the slow oxidation of organic substances in the soil. The di- and monocalcium phosphates react with sodium carbonate and bicarbonate and neutralise their alkaline property.



Consequently, the harmful effects of alkali carbonates are minimised due to the formation of sodium phosphate. Simultaneously,  $\text{Ca}(\text{HCO}_3)_2$  produced in the system supplies soluble calcium ions along with monocalcium phosphate. They slowly replace sodium ions from the exchange complex of the alkali soil, thereby leading to its reclamation.

Moreover, in the nitrification of proteins and nitrogenous compounds present in legumes and other plant materials, nitrous and nitric acids are formed which are profitable in the reclamation of alkali soils.

Keeping in view the above concept of the importance of organic matter and calcium phosphates in the reclamation of alkali soils, experiments were carried on in dishes in the laboratory by using different organic substances alone or mixed with rock phosphates and basic slags. These experiments were performed in the presence and absence of light in order to study the effect of light on nitrogen transformation. The organic substances used were starch, waterhyacinth and cactus and the sources of phosphates were German basic slag, Tata basic slag and Bihar rock phosphate.

### Experimental Procedure

200 gm of well powdered alkali soil obtained from Phulpur. (Allahabad) was taken in enamelled dishes and mixed thoroughly with different organic materials and phosphate. The rates of application of organic materials and phosphates were 1% and 0.5%  $P_2O_5$  respectively. The contents of dishes were thoroughly mixed and 50% moisture was maintained by adding distilled water for the first 15 days and then the level of moisture content was reduced to 25% in order to bring about proper aeration. In this way, two sets of experiments were performed under similar conditions. One set of experiments was exposed to artificial light from a 100 watt electric bulb while the other set was covered with thick black cloth to cut off light. Temperature was recorded daily.

At the time of analysis the contents of the dishes were thoroughly mixed and samples were taken and analysed for organic carbon, total nitrogen, exchangeable calcium, available  $P_2O_5$ , pH and electrical conductivity. Carbon was estimated by the method of Robinson, Mclean and Williams<sup>2</sup> and total nitrogen by the salicylic acid reduction method<sup>3</sup>. Hissink's method<sup>4</sup> was used for the determination of exchangeable calcium and available phosphate was estimated by Dyer's method<sup>5</sup>. pH was determined by glass electrode Beckman pH meter<sup>6</sup> and conductivity was determined by Leeds and Northrup Pt-Ir drum type bridge<sup>7</sup>.

### Results

#### *Analysis of the alkali soil obtained from Phulpur, Allahabad*

% analysis		% analysis	
Moisture	.. 1.210	Total nitrogen	.. 0.0410
Loss on ignition	.. 2.820	NH <sub>3</sub> -N	.. 0.0023
HCl insoluble	.. 84.190	NO <sub>3</sub> -N	.. 0.0032
Sesquioxide	.. 7.050	Total available N	.. 0.0055
Fe <sub>2</sub> O <sub>3</sub>	.. 4.1012	pH	.. 9.3
CaO	.. 1.2502	Conductivity	.. 11.85 m mhos/cm
MgO	.. 1.2400	Total bacteria count	.. 0.07 million/gm
Total P <sub>2</sub> O <sub>5</sub>	.. 0.1350	Total Azotobacter count	.. 0.04 million/gm
K <sub>2</sub> O	.. 0.7920	Fungus count	.. 0.012 million/gm
Available P <sub>2</sub> O <sub>5</sub>	.. 0.0280	Exchangeable Ca.m.e.	.. 4.8
Total carbon	.. 0.2205		

*Percentage Analysis of the organic substances*

	Starch	Cactus (Phylloclades)	Water hyacinth
Loss on ignition	...	80.3285	70.20
Ash	...	19.5712	29.780
HCl insoluble	...	2.4228	...
Sesquioxide	...	0.6638	...
Fe <sub>2</sub> O <sub>3</sub>	...	00.2320	11.120
CaO	...	11.9763	2.7450
MgO	...	1.1828	3.1645
K <sub>2</sub> O	...	2.2102	0.6378
P <sub>2</sub> O <sub>5</sub>	...	0.6325	0.6521
Total carbon	35.10	26.6900	26.8350
Total Nitrogen	...	1.6125	1.4563

*Percentage analysis of phosphatic materials*

	German basic slag	Tata basic slag	Bihar rock Phosphates
Silica	22.7525	22.56	20.6240
Sesquioxide	19.7520	23.850	29.5420
Fe <sub>2</sub> O <sub>3</sub>	16.2070	17.79	3.7542
Al <sub>2</sub> O <sub>3</sub>	3.5450	16.0250	25.7878
Total P <sub>2</sub> O <sub>5</sub>	17.5050	7.570	17.550
Available P <sub>2</sub> O <sub>5</sub>	10.0565	4.250	2.4725
CaO	33.5565	22.4560	11.0550
K <sub>2</sub> O	0.4525	8.6560	3.250
MgO	5.5050	4.0560	0.7505

TABLE 1  
200 gm. Soil + 1% C as starch

Average Temp. 30°C

Period of exposure in days	Organic carbon (gm)	Total nitrogen (gm)	Efficiency	Exchange-able Ca. (m.e.%)	Available P <sub>2</sub> O <sub>5</sub> (gm)	Electrical conductivity in m.hos/cm. (25°C)	pH
<i>Light</i>							
0	2.4410	0.0820	...	4.80	0.0560	11.85	9.3
30	1.6233	0.1016	23.9	6.72	0.0745	10.82	8.65
60	1.3670	0.1080	24.2	9.46	0.0885	10.30	8.40
90	1.0936	0.1129	22.9	11.86	0.1024	9.85	8.25
120	0.8983	0.1113	...	13.66	0.1180	9.46	8.05
<i>Dark</i>							
0	2.4410	0.0820	...	4.80	0.0560	11.85	9.3
30	1.7576	0.0912	13.4	5.42	0.0650	11.36	8.85
60	1.5574	0.0940	13.5	7.06	0.0732	11.20	8.70
90	1.3426	0.0957	12.4	8.26	0.0810	11.00	8.60
120	1.1229	0.0951	...	9.55	0.0890	10.80	8.45

TABLE 2  
200 gm. Soil + 1% C as starch + 0.5% P<sub>2</sub>O<sub>5</sub> as German basic slag

<i>Light</i>							
0	2.4410	0.0820	...	4.80	0.3432	11.85	9.3
30	1.3670	0.1233	38.4	8.25	0.3790	9.90	8.40
60	1.1717	0.1312	38.7	12.62	0.4115	8.65	7.65
90	0.9081	0.1396	37.5	15.92	0.4430	7.76	7.60
120	0.6835	0.1390	...	19.02	0.4705	6.75	7.40
<i>Dark</i>							
0	2.4410	0.0820	...	4.80	0.3432	11.85	9.3
30	1.5745	0.1024	23.5	6.65	0.3620	10.48	8.65
60	1.4500	0.1055	23.7	8.42	0.3785	9.72	8.35
90	0.1913	0.1101	22.4	10.23	0.3995	9.25	7.95
120	0.9398	0.1099	...	11.45	0.4103	8.80	7.90

TABLE 3  
200 gm. Soil + 1% C as starch + 0.5% P<sub>2</sub>O<sub>5</sub> as Tata basic slag

Average Temp. 30°C

Period of exposure in days	Organic Carbon (gm)	Total nitrogen (gm)	Efficiency	Exchange-able Ca. (m.e.%)	Avail-able P <sub>2</sub> O <sub>5</sub> (gm)	Electrical conductivity in m.mhos/cm. (25°C)	pH
<i>Light</i>							
0	2.4410	0.0820	...	4.8	0.3367	11.85	9.3
30	1.4012	0.1215	37.9	7.82	0.3715	10.05	8.45
60	1.1961	0.1296	38.2	12.01	0.4008	8.80	7.75
90	0.9398	0.1369	36.5	15.35	0.4291	7.82	7.70
120	0.7323	0.1362	...	18.50	0.4562	6.90	7.50
<i>Dark</i>							
0	2.4410	0.0820	...	4.8	0.3367	11.85	9.3
30	1.6013	0.1014	23.1	6.38	0.3492	10.00	8.65
60	1.4646	0.1049	23.4	8.20	0.3647	9.78	8.40
90	1.2157	0.1087	21.7	9.85	0.3797	9.34	8.00
120	0.9569	0.1085	...	11.02	0.3947	8.87	7.95

TABLE 4  
200 gm. Soil + 1% C as starch + 0.5% P<sub>2</sub>O<sub>5</sub> as Bihar rock phosphate

<i>Light</i>							
0	2.4410	0.0820	..	4.80	0.1264	11.85	9.3
30	1.4988	0.1151	35.1	7.22	0.1573	10.57	8.55
60	1.2913	0.1226	35.3	11.42	0.1834	9.06	8.25
90	1.0448	0.1290	33.6	14.15	0.2088	8.15	7.75
120	0.8300	0.1280	...	16.75	0.2340	7.33	7.55
<i>Dark</i>							
0	2.4410	0.0820	...	4.80	0.1264	11.85	9.3
30	1.6795	0.0976	20.5	6.02	0.1362	10.71	8.75
60	1.5428	0.1006	27.7	7.88	0.1496	9.99	8.60
90	1.3060	0.1041	19.4	8.85	0.1618	9.62	8.25
120	1.0350	0.1037	..	9.54	0.1739	9.21	8.05

TABLE 5  
200 gm. Soil + 1% C as water hyacinth

Average Temp. 28°C

Period of exposure in days	Organic carbon (gm)	Total nitrogen (gm)	Efficiency	Exchange-able Ca. (m.e. %)	Avail-able P <sub>2</sub> O <sub>5</sub> (gm)	Electrical conductivity in m.mhos/cm (25°C)	pH
<i>Light</i>							
0	2.4410	0.1872	...	4.8	0.0560	11.85	9.3
60	1.6721	0.2046	22.3	7.15	0.0720	10.85	8.65
120	1.4524	0.2094	22.4	10.20	0.0862	10.32	8.45
180	1.2572	0.2131	21.9	12.70	0.1002	9.90	8.25
240	1.0741	0.2105	...	14.30	0.1163	9.50	8.15
<i>Dark</i>							
0	2.4410	0.1872	...	4.8	0.0560	11.85	9.3
60	1.8284	0.1947	12.3	5.90	0.0660	11.40	8.85
120	1.6111	0.1974	12.3	7.60	0.0745	11.25	8.75
180	1.4598	0.1992	12.3	8.90	0.0821	11.05	8.65
240	1.3035	0.1980	...	10.20	0.0902	10.92	8.50

TABLE 6  
200 gm. Soil + 1% C as Water hyacinth + 0.5% P<sub>2</sub>O<sub>5</sub> as German basic slag

<i>Light</i>							
0	2.4410	0.1872	...	4.8	0.3432	11.85	9.3
60	1.4598	0.2230	36.5	9.1	0.3757	10.10	8.45
120	1.2547	0.2309	36.8	13.2	0.4080	8.80	7.70
180	1.0741	0.2370	36.4	16.7	0.4400	7.80	7.65
240	0.8666	0.2361	...	20.0	0.4720	7.00	7.50
<i>Dark</i>							
0	2.4410	0.1872	...	4.8	0.3432	11.85	9.3
60	1.6404	0.2049	22.2	7.2	0.3599	10.55	8.65
120	1.4842	0.2087	22.4	8.9	0.3764	9.78	8.35
180	1.2888	0.2127	22.1	10.6	0.3927	9.30	8.00
240	1.0619	0.2123	...	12.0	0.4088	8.85	7.90

TABLE 7

Average Temp. 28°C

200 gm. Soil + 1% C as water hyacinth + 0.5% P<sub>2</sub>O<sub>5</sub> as Tata basic slag

Period of exposure in days	Organic carbon (gm)	Total nitrogen (gm)	Efficiency	Exchange-able Ca. (m.e. %)	Avail-able P <sub>2</sub> O <sub>5</sub> (gm)	Electrical conductivity in m.mhos/cm 25°C	pH
<i>Light</i>							
0	2.4410	0.1872	...	4.8	0.3367	11.85	9.3
60	1.4720	0.2216	35.6	8.6	0.3685	10.20	8.50
120	1.2718	0.2291	35.8	12.9	0.3978	8.88	7.75
180	1.0985	0.2349	35.5	16.3	0.4263	7.85	7.65
240	0.8910	0.2338	...	19.6	0.4546	7.10	7.50
<i>Dark</i>							
0	2.4410	0.1872	...	8.4	0.3367	11.85	9.3
60	1.6502	0.2043	21.7	6.8	0.3472	10.60	8.70
120	1.4988	0.2079	21.9	8.7	0.3629	9.82	8.40
180	1.3109	0.2119	21.8	10.4	0.3781	9.35	8.10
240	1.0741	0.2114	...	11.6	0.3932	8.90	7.95

TABLE 8

200 gm. Soil + 1% C as water hyacinth + 0.5% P<sub>2</sub>O<sub>5</sub> as Bihar rock phosphate

<i>Light</i>							
0	2.4410	0.1872	...	4.8	0.1264	11.85	9.3
60	1.5818	0.2152	32.7	8.2	0.1548	10.60	8.60
120	1.3426	0.2228	32.9	12.0	0.1810	9.12	8.30
180	1.1766	0.2282	32.4	14.7	0.2064	8.22	7.80
240	0.9716	0.2267	...	18.4	0.2317	7.40	7.60
<i>Dark</i>							
0	2.4410	0.1872	...	4.80	0.1264	11.85	9.3
60	1.7576	0.1997	18.3	6.4	0.1347	10.73	8.80
120	1.5818	0.2030	18.4	8.2	0.1482	10.02	8.65
180	1.3914	0.2067	18.3	9.1	0.1605	9.65	8.30
240	1.1376	0.2059	...	10.3	0.1727	9.25	8.15

TABLE 9  
200 gm. Soil + 1% C as Cactus

Average Temp. 28°C

Period of exposure in days	Organic carbon (gm)	Total nitrogen (gm)	Efficiency	Exchange-able Ca. (m.e. %)	Avail-able P <sub>2</sub> O <sub>5</sub> (gm)	Electrical conductivity in m.mhos/cm. (25°C)	pH
<i>Light</i>							
0	2.4410	0.2028	...	4.8	0.0560	11.85	9.3
60	1.7551	0.2175	21.5	7.05	0.0682	10.90	8.65
120	1.5184	0.2227	21.5	10.1	0.0823	10.46	8.50
180	1.3377	0.2263	21.2	12.5	0.0964	10.00	8.35
240	1.1180	0.2239	...	14.1	0.1125	9.75	8.25
<i>Dark</i>							
0	2.4410	0.2028	...	4.8	0.0560	11.85	9.3
60	1.8845	0.2093	11.8	5.85	0.0622	11.51	8.85
120	1.7039	0.2113	11.5	7.5	0.0702	11.32	8.75
180	1.5232	0.2134	11.5	8.6	0.0784	11.16	8.65
240	1.3670	0.2124	...	9.9	0.0865	11.02	8.50

TABLE 10  
200 gm. Soil + 1% C as Cactus + 0.5% P<sub>2</sub>O<sub>5</sub> as German basic slag

<i>Light</i>							
0	2.4410	0.2028	...	4.8	0.3432	11.85	9.3
60	1.5232	0.2358	36.0	8.9	0.3734	10.18	8.45
120	1.2889	0.2446	36.0	13.0	0.4034	8.92	7.75
180	1.1107	0.2508	36.0	16.6	0.4332	7.90	7.65
240	0.9398	0.2500	...	19.8	0.4628	7.10	7.55
<i>Dark</i>							
0	2.4410	0.2028	...	4.8	0.3432	11.85	9.3
60	1.7234	0.2184	21.8	7.05	0.3589	10.58	8.65
120	1.4964	0.2235	21.9	8.85	0.3745	9.85	8.35
180	1.3035	0.2279	22.0	10.60	0.3899	9.35	8.10
240	1.1595	0.2276	...	11.8	0.4052	8.92	8.05



TABLE 11  
200 gm. Soil + 1% C as Cactus + 0.5% P<sub>2</sub>O<sub>5</sub> as Tata basic slag

Period of exposure in days	Organic carbon (gm)	Total nitrogen (gm)	Efficiency	Exchangeable Ca. (m.e. %)	Available P <sub>2</sub> O <sub>5</sub> (gm)	Electrical conductivity in m.mhos/cm. 25°C	pH
<i>Light</i>							
0	2.4410	0.2028	...	4.8	0.3367	11.85	9.3
60	1.5428	0.2344	35.2	8.48	0.3625	10.28	8.55
120	1.3109	0.2430	35.5	12.7	0.3918	8.98	7.75
180	1.1400	0.2487	35.2	16.1	0.4205	8.02	7.65
240	0.9716	0.2477	...	19.0	0.4488	7.25	7.55
<i>Dark</i>							
0	2.4410	0.2028	...	4.8	0.3367	11.85	9.35
60	1.7405	0.2177	21.4	6.7	0.3472	10.68	8.7
120	1.5135	0.2227	21.4	8.65	0.3629	10.00	8.45
180	1.3255	0.2268	21.5	10.10	0.3781	9.40	8.15
240	1.1791	0.2264	...	11.4	0.3932	9.05	8.05

TABLE 12  
200 gm. Soil + 1% C as Cactus + 0.5% P<sub>2</sub>O<sub>5</sub> as Bihar rock phosphate

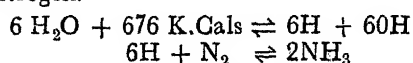
<i>Light</i>							
0	2.4410	0.2028	...	4.8	0.1264	11.85	9.3
60	1.6550	0.2283	32.5	8.15	0.1498	10.90	8.60
120	1.3963	0.2372	32.9	11.85	0.1762	9.25	8.35
180	1.2254	0.2427	32.8	14.60	0.2016	8.56	7.85
240	1.0497	0.2413	...	17.40	0.2270	8.00	7.70
<i>Dark</i>							
0	2.4410	0.2028	...	4.8	0.1264	11.85	9.3
60	1.8381	0.2137	18.1	6.3	0.1347	10.82	8.80
120	1.5867	0.2185	18.4	8.18	0.1482	10.25	8.65
180	1.4012	0.2218	18.2	9.05	0.1605	9.85	8.35
240	1.2450	0.2211	...	10.20	0.1727	9.60	8.25

## Discussion

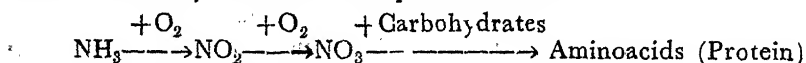
From the foregoing observations it is clear that carbonaceous materials like starch, water hyacinth and cactus, when added to the alkali soils, are slowly oxidised in contact with air and there is a considerable increase in the total nitrogen content. It will be observed in these tables (Tables 1 to 12) that the amount of carbon oxidised in presence of light is greater than the amount oxidised in the dark. Also, the efficiency of nitrogen fixation, i.e. the amount of nitrogen fixed in milligram per gram of carbon oxidised, is greater in presence of light than in its absence with all the three types of organic substances studied. The following order of the rate of oxidation of organic materials has been noted :

Starch > Water hyacinth > Cactus.

The same order of the rate of oxidation follows in the sets kept in the dark, although, the amount of carbon oxidised in the dark set is less. In explaining the nitrogen fixation in soils, Dhar postulates that the atomic hydrogen, formed during the decomposition of water by the energy liberated from oxidation of carbonaceous materials in the soil, combines with the atmospheric nitrogen and thus fixes nitrogen.



The ammonia formed readily undergoes oxidation and forms nitrite and nitrate and finally aminoacids and proteins in the soil.



The reaction is enhanced in sunlight or artificial light, Ranjan and Bhattacharya<sup>8</sup> also observed photochemical nitrogen fixation in sterile soils using cellulose as energy material. Kerr<sup>9</sup> has also shown that the fixation of nitrogen in light is double than in dark in some of the sandy loamy soils.

It is interesting to note that the rate of oxidation of organic materials and efficiency of nitrogen fixation increases in the sets where different phosphatic amendments like German basic slag, Tata basic slag and Bihar rock phosphate have been mixed with organic materials. It has been observed that the oxidation of carbon from the various materials is greater with basic slag than with rock phosphate. This is due to the fact that basic slag is more alkaline than rock phosphate and it is well known that alkalinity accelerates the oxidation. The order of efficiency of basic slags and rock phosphates in oxidising organic matter is as follows :

German basic slag > Tata basic slag > Bihar rock phosphate

The order of efficiency of the nitrogen fixation of different organic matter with basic slags and rock phosphate is as follows :

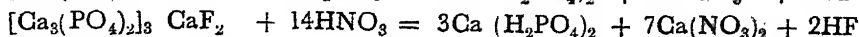
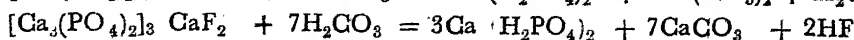
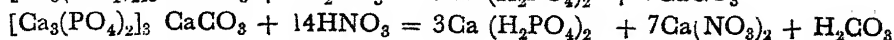
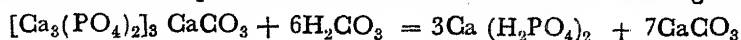
Starch > Water hyacinth > Cactus

It is also interesting to note from the experimental results (Table 1 to Table 12) that loss of nitrogen takes place after 90 days in the case of starch and after 180 days in the case of water hyacinth and cactus. This is due to the fact that the C/N ratio gradually tends to decrease. However, it has been observed that the loss of nitrogen is retarded in the sets where phosphate amendments have been added. This is due to the formation of stable phospho-proteins by the reaction of proteins and phosphorus compounds. Similar results have been obtained by Dhar, Mitra and Shanker<sup>10</sup>.

The results recorded in Tables 1-12 show that the exchangeable calcium of the system increases when different phosphatic amendments are added to alkali soil in presence of different organic substances. The increase is greater in light than in the dark. The increase in exchangeable calcium observed on the addition of phosphatic amendments along with organic matter, is much greater than on the addition of organic matter alone. In this case, carbonic acid and organic acids, produced by the decomposition of organic matter and small amount of nitric acid produced by the nitrification of organic compounds dissolve not only the native soil phosphates from alkali soil but also dissolve phosphates from phosphatic amendments. Soluble calcium ions thus brought into solution replace sodium ions from the exchange complex of the soil thereby increasing the exchangeable calcium status of the system. Lyon and Buckman<sup>11</sup> have suggested that the following reaction takes place on the application of basic slag to soils :

$(\text{CaO})_5 \text{P}_2\text{O}_5\text{SiO}_2 + 8\text{CO}_2 + 6\text{H}_2\text{O} = \text{Ca}(\text{H}_2\text{PO}_4)_2 + 4\text{Ca}(\text{HCO}_3)_2 + \text{SiO}_2$   
The soluble  $\text{Ca}(\text{H}_2\text{PO}_4)_2$  and  $\text{Ca}(\text{HCO}_3)_2$  are effective in the increase of exchangeable calcium.

The chief phosphatic constituents of rock phosphate, carbonato-apatite  $[\text{Ca}_3(\text{PO}_4)_2]_3 \text{CaCO}_3$  and fluorapatite  $[\text{Ca}_3(\text{PO}_4)_2]_3 \text{CaF}_2$  react with carbonic acid and nitric acids produced in the alkali soils in the following manner :



The soluble  $\text{Ca}(\text{H}_2\text{PO}_4)_2$  and  $\text{Ca}(\text{NO}_3)_2$  produced supply calcium ions for the improvement of exchangeable calcium status of alkali soils.

It is noted from Tables 1 to 12 that the available phosphate of the system increases with the oxidation of organic matter and that a greater increase of available phosphate is obtained by the addition of phosphatic amendments along with organic matter to alkali soil than when organic matter alone is added. This enhanced availability can be ascribed to the soluble phosphates obtained from the phosphatic amendments. With the oxidation of organic matter carbonic acid and organic acids are produced which are responsible for the production of di- and small amounts of mono-calcium phosphates. Moreover, the organic acids like oxalic, propionic, citric etc. prevent phosphate precipitation by iron and aluminium and thus increase the availability of phosphates. Similarly, nitrous and nitric acids formed in the ammonification and nitrification of proteins make phosphate more readily available. Bauer<sup>12</sup>, Copeland and Merkle<sup>13</sup> Gerretsen<sup>14</sup> and Midgley and Dunklee<sup>15</sup> have shown that phosphate added to the soil with organic matter has greater availability than the same amount of phosphate added alone. Bradley and Sieling<sup>16</sup>, Struther and Sieling<sup>17</sup> and Swenson, Cole and Sieling<sup>18</sup> have also shown the organic acids produced by the oxidation of organic matter in soil increase the availability of phosphate by preventing its precipitation by iron and aluminium. Jensen<sup>19</sup> reported that water extracts of decomposing organic matter were 1 to 5 times more effective than water alone in dissolving phosphate from soils.

The results recorded in Tables 1 to 12 show a decrease in electrical conductivity with the oxidation of organic matter. The different acids (already described), produced by the oxidation of organic matter, react with the alkalinity of the alkali soil and form water and neutral salts. It is well known that the

electrical conductivity of sodium hydroxide is much greater than the electrical conductivity of sodium salts like sodium chloride, sodium sulphate under equivalent concentrations. Hence, in the process of decrease of the pH values of alkali soils, sodium salts are produced at the expense of sodium hydroxide which is always present in the alkali soils and thus the electrical conductivity of the water extract decreases. This view can be supported by the work done by Davies<sup>20</sup> and Glasstone<sup>21</sup>, Harned and Reven<sup>22</sup> and Mc Innes<sup>23</sup>.

From a study of Tables 1 to 12, it is clear that the lowering of pH on the addition of phosphatic amendments along with organic matter is greater than when organic matter alone is added. This is because of the fact that phosphate increases the rate of oxidation of organic matter there by producing more carbonic acid and organic acids like oxalic, citric, malic etc. which bring down the pH. Moreover, nitrous and nitric acids formed in the ammonification and nitrification of proteins contained in organic materials also neutralise the alkalinity. Thus, the rate of oxidation of organic matter is directly proportional to the lowering of pH value.

Thus, it is clear from these investigations that the addition of a mixture of organic matter and phosphate to alkali soil brings about a permanent reclamation by increasing total nitrogen, exchangeable calcium, available phosphate and humus status and by decreasing pH value and electrical conductivity of soil.

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## An Extension of a Theorem of Lehmer

By

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D. H. Lehmer proved [1] the following result.

Let  $m, r, k, h, q_0, q_1, \dots, q_k$  be integers such that  $m \geq 2, 0 \leq h \leq k, q_i \geq 0$  ( $i = 0, 1, \dots, k$ ). Then the sum

$$\sum (a_0 q_0 + a_1 q_1 + \dots + a_k q_k)^h$$

taken over all sets  $(a_0, a_1, \dots, a_k)$  satisfying

$$0 \leq a_i \leq m-1 \quad (0 \leq i \leq k)$$

and

$$a_0 + a_1 + \dots + a_k \equiv r \pmod{m},$$

is independent of  $r$ .

The object of the present note is to indicate an extension of this result. The method of proof is similar to that of Lehmer except that it makes use of 'Operator's rather than differentiation. A similar extension has been obtained by me in [2] in a slightly less general form by using the multinomial theorem. We shall prove here

**Theorem.** Let  $q_0, q_1, \dots, q_k$  be an arbitrary set of  $k+1$  complex numbers (distinct or not), and let  $m$  be an integer  $\geq 2$ .

Let  $S$  be the collection of all numbers of the form

$$(1) \quad n = a_0 q_0 + a_1 q_1 + \dots + a_j q_j + \dots + a_k q_k$$

and  $S(j)$  the collection of all numbers of the form

$$(2) \quad n_j = a_0 q_0 + a_1 q_1 + \dots + (m-1-a_j) q_j + \dots + a_k q_k, \quad (0 \leq j \leq k)$$

where the  $a_i$  are integers satisfying

$$0 \leq a_i \leq m-1 \quad (0 \leq i \leq k)$$

Further, let  $S_r, 0 \leq r \leq m-1$ , be the collection of elements of  $S$ , and  $S_r(j), 0 \leq j \leq k$ , the collection of all elements of  $S(j)$  for each of which

$$a_0 + a_1 + \dots + a_k \equiv r \pmod{m}$$

Then for  $P(x)$ , a complex polynomial of degree less than or equal to  $k$ , each of the sums  $\sum_{n \in S_r} P(x+n)$  and  $\sum_{n_j \in S_r(j)} P(x+n_j), 0 \leq r \leq m-1, 0 \leq j \leq k$ , is independent of  $r$ ,

and moreover  $\sum_{n \in S_r} P(x+n) = \sum_{n_j \in S_r(j)} P(x+n_j)$

*Proof of the theorem.* Let  $b$  be an arbitrary complex number and let  $E(b)$  be an operator which maps the polynomial  $P(x)$  onto the polynomial  $P(x+b)$ . Then since

$$E(a) E(b) P(x) = E(a) P(x+b) = P(x+b+a) = P(x+a+b) \text{ and } E(a+b) P(x) = P(x+a+b) \text{ we have}$$

$$(3) \quad E(a+b) = E(a) E(b)$$

In what follows  $m$  is to be a fixed integer  $\geq 2$  and  $e$  any  $m$ th root of unity. We first prove that  $\sum_{n \in S_r} P(x+n)$  is independent of  $r$ ,  $0 \leq r \leq m-1$ . Consider the

product

$$\prod_{i=0}^k [e^0 E(0 q_i) + e^1 E(1 q_i) + \dots + e^{m-1} E(m-1 q_i)]$$

On multiplying out and using (3) we get the following algebraic identity

$$(4) \quad \prod_{i=0}^k \sum_{r=0}^{m-1} e^r E(r q_i) = \sum_{n \in S} e^{v(n)} E(n)$$

where  $v(n) = a_0 + a_1 + \dots + a_k$  when  $n$  is given by (1); and further if a number  $n \in S$  has more than one representation of the specified kind, the right hand sum in (4) includes a term for each representation. If we now take  $e \neq 1$  we find that

$$\begin{aligned} \sum_{r=0}^{m-1} e^r E(r q_i) P(x) &= \sum_{r=0}^{m-1} e^r P(x+r q_i) \\ &= c_q x^q \sum_{r=0}^{m-1} e^r + \sum_{r=0}^{m-1} e^r Q(x, r) \end{aligned}$$

where  $c_q$  is the leading coefficient of  $P(x)$  and  $Q(x, r)$  is a polynomial of degree smaller than  $P(x)$ . Then since

$$(5) \quad \sum_{r=0}^{m-1} e^r = 0$$

we find that

$$\sum_{r=0}^{m-1} e^r E(r q_i), e \neq 1$$

maps a polynomial  $P(x)$  onto a polynomial of smaller degree. We conclude from this result that when  $P(x)$  has degree  $\leq k$ , the operator on the left side of the equation (4) maps  $P(x)$  onto zero. Hence by (4)

$$(6) \quad \sum_{n \in S} e^{v(n)} E(n) P(x) = \sum_{n \in S} e^{v(n)} P(x+n)$$

$$= \sum_{r=0}^{m-1} e^r \sum_{n \in S_r} P(x+n) = 0, e \neq 1.$$

We have to show that for all polynomials  $P(x)$  of degree  $\leq k$

$$(7) \quad \sum_{n \in S_0} P(x+n) = \dots = \sum_{n \in S_{m-1}} P(x+n)$$

If we denote by  $D_r$  the difference

$$D_r = \sum_{n \in S_r} P(x+n) - \sum_{n \in S_0} P(x+n)$$

it suffices to show that

$$(8) \quad D_r = 0 \quad (r = 1, \dots, m-1)$$

In view of (5) and (6) we have

$$(9) \quad \sum_{r=1}^{m-1} e^r D_r = 0$$

Now (9) holds for every  $m$ th root of unity  $e \neq 1$ . Hence if we consider a primitive  $m$ th root of unity, the other  $e$ 's are of the form  $e^q$  ( $q = 2, 3, \dots, m-1$ ). Thus (9) gives rise to a set of  $m-1$  homogeneous equations

$$(10) \quad \sum_{r=1}^{m-1} e^{qr} D_r = 0 \quad (q = 1, 2, \dots, m-1)$$

in  $D_1, D_2, \dots, D_{m-1}$  with matrix  $(e^{qr})$ ,  $1 \leq q \leq m-1, 1 \leq r \leq m-1$ . This matrix is of Vandermonde type in distinct numbers  $e, e^2, \dots, e^{m-1}$  and hence the matrix is non-singular. Therefore the equations (10) has no solutions other than  $D_r = 0, r = 1, 2, \dots, m-1$ . Thus (8) and hence (7) is proved.

We next prove that  $\sum_{n_j \in S_r(j)} P(x+n_j)$  is independent of  $r, 0 \leq r \leq m-1$ ,

$0 \leq j \leq k$ . Here we construct, corresponding to (4), the following identity for any fixed  $j = 0, 1, \dots, k$ :

$$(11) \quad \sum_{r=0}^{m-1} e^r E(rq_0) \sum_{r=0}^{m-1} e^r E(rq_1) \dots \sum_{r=0}^{m-1} e^r E(\overline{m-1-r} q_j) \dots \sum_{r=0}^{m-1} e^r E(rq_k) \\ = \sum_{n_j \in S(j)} e^{v(n_j)} E(n_j)$$

where  $v(n_j) = a_0 + \dots + a_k$  when  $n_j$  is given by (2). As before we see that the operator on the left side of (11) maps a polynomial  $P(x)$  of degree  $\leq k$  onto zero. Hence

$$\begin{aligned}
 (12) \quad \sum_{n_j \in S(j)} e^{v(n_j)} E(n_j) P(x) &= \sum_{n_j \in S(j)} e^{v(n_j)} P(x+n_j) \\
 &= \sum_{r=0}^{m-1} e^r \sum_{n_j \in S_r(j)} P(x+n_j) = 0, e \neq 1
 \end{aligned}$$

and the proof that  $\sum_{n_j \in S_r(j)} P(x+n_j)$  is independent of  $r$  follows as before. Finally since from (6) and (12)

$$\sum_{n \in S_r} P(x+n) \sum_{r=0}^{m-1} e^r = \sum_{n_j \in S_r(j)} P(x+n_j) \sum_{r=0}^{m-1} e^r = 0$$

for every  $m$ th roots of unity  $e \neq 1$ , we have, for  $P(x)$  a polynomial of degree  $\leq k$

$$(13) \quad \sum_{n \in S_r} P(x+n) = \lambda \sum_{n_j \in S_r(j)} P(x+n_j); 0 \leq r \leq m-1, 0 \leq j \leq k,$$

where  $\lambda > 0$  is a constant. But as  $a_j$  runs through a complete set of residues (mod  $m$ ) so does  $m-1-a_j$ . Thus if we replace  $a_j$  by  $m-1-a_j$  in each side of (13), we should have

$$(14) \quad \sum_{n_j \in S_r(j)} P(x+n_j) = \sum_{n \in S_r} P(x+n)$$

From (13) and (14) we have  $\lambda = 1$  and so  $\sum_{n \in S_r} P(x+n) = \sum_{n_j \in S_r(j)} P(x+n_j)$ . This completes the proof of the theorem.

Taking the  $q$ 's positive integers,  $P(x) = x^h$ ,  $0 \leq h \leq k$  and  $x = 0$  in the theorem, we have that the  $m(k+2)$  sets,  $S_r, S_r(j)$ ,  $0 \leq r \leq m-1$ ,  $0 \leq j \leq k$ , each with  $m^k$  elements, constitute a solution of the Tarry-Escott problem of degree  $k$  and order  $m(k+2)$ . Of these  $m(k+2)$  sets, the  $m$  sets  $S_r$  ( $0 \leq r \leq m-1$ ) are the Lehmer's, which afford a solution of the Tarry-Escott problem of degree  $k$  and order  $m$  in accordance with his result stated in the very beginning of this paper. In [2] we constructed a numerical example with  $m = 3$ ,  $k = 2$ ,  $q_0 = 1$ ,  $q_1 = 2$ ,  $q_2 = 5$  for this extended case. We also gave there the corresponding extension of Prouhet's rule [3].

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# Transition Probabilities for Electronic Spectra of $\text{Li}_2$ and $\text{C}_2$

By

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The computations of transition probabilities in diatomic and polyatomic molecules have made use of the dipole moment operator exclusively in its dipole length form<sup>1</sup>. It is well known, however that the dipole operator may be expressed in other formally equivalent forms, namely, in the velocity and acceleration forms. Calculations using the dipole velocity operator for atoms have been made by many workers<sup>2-4</sup>.

The detail of the formula for the dipole length operator method of calculating transition probabilities, may be found in Mulliken and Rieke's review<sup>1</sup>. The dipole strength  $D$  for a transition between two electronic states,  $k$  and  $l$ , is given by

$$D = \sum_i G Q_i^2 \quad (1)$$

in which

$$Q_i = \int \psi_k q_i \psi_l d\tau \quad (2)$$

$q_i$  being the vector  $x, y, z$ . We are interested in calculating the oscillator strength for the transition between the states  $k$  and  $l$  (classically, the number of dispersion electrons involved in the transition). This oscillator strength is given by the following formula derived for a single narrow absorption band at wave number  $\nu$  in the gaseous state.

$$f = (8\pi^2 mc/3 h) \nu D = 1.085 \times 10^{11} \nu D \quad (3)$$

(for  $\nu$  in  $\text{cm}^{-1}$  and  $D$  in  $\text{cm}^2$ )

Previous practice has made use of this formula calculating  $D$  theoretically by Eq's. (1) and (2), and inserting the observed frequency of the transition for  $\nu$ . This, is then, what may be called a semitheoretical treatment.

The calculation of oscillator strengths making use of the dipole velocity operator uses Eq's. (1) and (3) in the form presented here but in place of Eq. (2) the equivalent integral

$$Q_i = [2/(E_l - E_k)] \int \psi_k \frac{\delta}{\delta q_i} \psi_l d\tau \quad (4)$$

is employed. It should be noted that the energy difference  $E_l - E_k$ , in Eq. (4) is given in the units of Rydbergs. If we absorb the Rydberg constant as well as the

conversion factor from *cm* to atomic units in Eq. (3), we arrive at the following convenient formula for *f*

$$f = 1.464 \times 10^5 G \left[ \int \psi_k \left( \sum_i \frac{\partial}{\partial q_i} \right) \psi_l d\tau \right]^2 / \nu \quad (5)$$

In this formula, the energy difference  $E_l - E_k$  has been used to cancel the frequency appearing in Eq. (3). For the purposes of the present calculation, the formula will be used in this form with substitution of the observed frequency. In equation (5),  $\nu$  is in the wave numbers and the integral is expressed in the atomic units. *G* is a factor which takes into account the number of independent final states to which transitions are possible from a single initial state. For the calculations presented here *G* is one for all the states of  $Li_2$  and  $C_2$ .

The following transitions  $Li_2$ ,  $A^1\Sigma_u^+ - X^1\Sigma_g^+$  and  $c^1\Sigma_u^+ - X^1\Sigma_g^+$ , and  $C_2$ ;  $b^1\pi_u - x^1\Sigma_g^+$ ,  $B^3\Pi_g - X^3\Sigma_u^-$ ,  $d^1\Sigma_u^+ - x^1\Sigma_g^+$ ,  $c^1\Pi_g - b^1\pi_u$ , and  $A^3\Pi_u - X^3\Pi_u$  have been studied here. Slater atomic orbitals have been used to form molecular orbital by the LCAO method. The constants determining the molecular orbitals have been taken from Ransil's<sup>5</sup> work. The method of calculation is similar to one used by Shull<sup>6</sup>. The values of  $\nu$  have been taken from Hergberg<sup>7</sup>. The results obtained are in good agreement, in the case of  $C_2$  for  $^3\Pi_g - ^3\Pi_u$  and  $^1\Pi_g - ^1\Sigma_u^+$  transition with the previous work of Shull<sup>6,8</sup>. Table 1 shows the value of the integrals and Table 2 shows the transition probabilities for different electronic transitions of  $Li_2$  and  $C_2$ .

TABLE I  
Values of Integrals Used

Integral	$Li_2$	$C_2$
$\int 1 s_a \frac{\partial}{\partial x} 1 s_b d\tau$	5.9824	13.4902
$\int 1 s_a \frac{\partial}{\partial x} 2s_b d\tau$	0.0313	0.0693
$\int 1 s_a \frac{\partial}{\partial x} 2p\sigma_a d\tau$	0.2567	0.7266
$\int 1 s_a \frac{\partial}{\partial x} 2p\sigma_b d\tau$	0.0518	0.1150
$\int 2 s_a \frac{\partial}{\partial x} 2s_b d\tau$	0.0398	0.0761
$\int 2 s_a \frac{\partial}{\partial x} 2p\sigma_a d\tau$	0.1876	0.4691
$\int 2 s_a \frac{\partial}{\partial x} 2p\sigma_b d\tau$	0.0347	0.1460
$\int 2p\sigma_a \frac{\partial}{\partial x} 2p\sigma_b d\tau$	-0.1434	-0.1566

TABLE 2  
Computed Transition Probabilities

Molecule	Transition	$\nu$	f
Li <sub>2</sub>	$2\sigma_g-2\sigma_u$ $A^1\Sigma_u^+-X^1\Sigma_g^+$	14068	0.8688
	$2\sigma_g-3\sigma_u$ $c^1\Sigma_u^+-X^1\Sigma_g^+$	30558	0.0158
C <sub>2</sub>	$1\pi_u-3\sigma_g$ $b^1\pi_u-x^1\Sigma_g$	8391	0.8931
	$1\pi_u-1\pi_g$ $B^3\Pi_g-X^3\Pi_u$	40080	0.0995
	$2\sigma_u-3\sigma_g$ $\left[ \begin{array}{l} d^1\Sigma_u^+-x^1\Sigma_g^+ \\ c^1\Pi_g-b^1\pi_u \\ A^3\Pi_g-X^3\Pi_u \end{array} \right.$	43240	0.1206
		25870	0.2016
		19306	0.2702

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## Neutron Yield in $\text{Li}^7(d,n)\text{Be}^8$

By

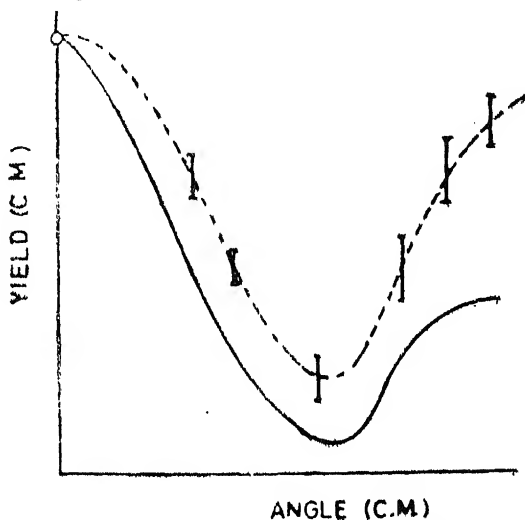
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The experimental neutron yield for the 2.8 Mev level of  $\text{Be}^8$  has been compared with the Butler's<sup>1</sup> theory (1957). A comparison of the data shows that the target lithium nucleus accepts a proton of angular momentum = 1 and the spin of this level is either 0 or 2 with even parity.

The neutrons from the deuteron (500 Kev) bombardment have been collected over  $400\mu$  thick Ilford nuclear Emulsion plates at  $0^\circ$ ,  $45^\circ$ ,  $60^\circ$ ,  $90^\circ$ ,  $120^\circ$ ,  $135^\circ$ , and  $150^\circ$ . The neutron density in centre of mass system after normalisation in forward direction shows a pronounced maximum and the fore-and-aft symmetry is missing. Data has been further subjected in light of Butler's theory.



The yield has been obtained using Butler's theory for various values of proton angular momentum transfer. The curve at  $l_p = 1$  (fig. 1) fits well among the curves that are drawn at  $l_p = 0, 1, 2$  and  $3$ . This suggests that the 2.8 Mev level of  $\text{Be}^8$  accepts a proton of angular momentum one directly and the Spin of this level is  $0^+$  or  $2^+$  (the odd values being not permissible for the experimentally observed break up of  $\text{Be}^8$  into two  $\alpha$ -particles (Crussards<sup>2</sup> and Jones *et al*<sup>3</sup>).

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